Engineering Mechanics: Statics

Course Overview

Engineering Mechanics
- Statics (Freshman Fall)
- Dynamics (Freshman Spring)
- Strength of Materials (Sophomore Fall)
- Mechanism Kinematics and Dynamics (Sophomore Spring)
- Aircraft structures (Sophomore Spring and Junior Fall)
- Vibration (Senior)

Statics: \[ \sum F = 0 \] force distribution on a system

Dynamics: \[ x(t) = f(F(t)) \] displacement as a function of time and applied force

Strength of Materials: \[ \delta = f(F) \] deflection of deformable bodies subject to static applied force

Vibration: \[ x(t) = f(F(t)) \] displacement on particles and rigid bodies as a function of time and frequency
Chapter 1 General Principles

- Basic quantities and idealizations of mechanics
- Newton’s Laws of Motion and gravitation
- Principles for applying the SI system of units

Chapter Outline

- Mechanics
- Fundamental Concepts
- The International System of Units
- Numerical Calculations
- General Procedure for Analysis
1.1 Mechanics

- Mechanics can be divided into:
  - Rigid-body Mechanics
  - Deformable-body Mechanics
  - Fluid Mechanics

- Rigid-body Mechanics deals with
  - Statics – Equilibrium of bodies; at rest or moving with constant velocity
  - Dynamics – Accelerated motion of bodies
1.2 Fundamentals Concepts

Basic Quantities
- Length - locate the position of a point in space
- Mass - measure of a quantity of matter
- Time - succession of events
- Force - a “push” or “pull” exerted by one body on another

Idealizations
- Particle - has a mass and size can be neglected
- Rigid Body - a combination of a large number of particles
- Concentrated Force - the effect of a loading
1.2 Newton’s Laws of Motion

- **First Law** - A particle originally at rest, or moving in a straight line with constant velocity, will remain in this state provided that the particle is not subjected to an unbalanced force.

- **Second Law** - A particle acted upon by an *unbalanced force* $F$ experiences an acceleration $a$ that has the same direction as the force and a magnitude that is directly proportional to the force.

- **Third Law** - The mutual forces of action and reaction between two particles are equal and, opposite and collinear.
Chapter 2  Force Vector

Chapter Outline

• Scalars and Vectors
• Vector Operations
• Addition of a System of Coplanar Forces
• Cartesian Vectors
• Position Vectors
• Force Vector Directed along a Line
• Dot Product
2.1 Scalar and Vector

- Scalar: A quantity characterized by a positive or negative number, indicated by letters in italic such as \( A \), e.g. Mass, volume and length

- Vector: A quantity that has magnitude and direction, e.g. position, force and moment, presented as \( A \) and its magnitude (positive quantity) as \( |A| \)

- Vector Subtraction \( R' = A - B = A + (-B) \)

Finding a Resultant Force

- *Parallelogram law* is carried out to find the resultant force \( F_R = (F_1 + F_2) \)
2.4 Addition of a System of Coplanar Forces

Scalar Notation: Components of forces expressed as algebraic scalars

\[ F_x = |F| \cos \theta \quad \text{and} \quad F_y = |F| \sin \theta \]

Cartesian Vector Notation in unit vectors \( \mathbf{i} \) and \( \mathbf{j} \)

\[ \mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} \]

- Coplanar Force Resultants

\[ \mathbf{F}_1 = F_{1x} \mathbf{i} + F_{1y} \mathbf{j} \]
\[ \mathbf{F}_2 = -F_{2x} \mathbf{i} + F_{2y} \mathbf{j} \]
\[ \mathbf{F}_3 = F_{3x} \mathbf{i} - F_{3y} \mathbf{j} \]
\[ \mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (F_{Rx}) \mathbf{i} + (F_{Ry}) \mathbf{j} \]

Scalar notation

\[ F_{Rx} = F_{1x} - F_{2x} + F_{3x} \]
\[ F_{Ry} = F_{1y} + F_{2y} - F_{3y} \]
Example 2.6

The link is subjected to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude and orientation of the resultant force.

Cartesian Vector Notation

$\mathbf{F}_1 = \{ 600\cos 30^\circ \ i + 600\sin 30^\circ \ j \} \ N$

$\mathbf{F}_2 = \{ -400\sin 45^\circ \ i + 400\cos 45^\circ \ j \} \ N$

Thus,

$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$

$\quad = (600\cos 30^\circ - 400\sin 45^\circ)\mathbf{i}$

$\quad \quad + (600\sin 30^\circ + 400\cos 45^\circ)\mathbf{j}$

$\quad = \{236.8\mathbf{i} + 582.8\mathbf{j}\} N$
2.5 Cartesian Vectors

- **Right-Handed Coordinate System**
  A right-handed rectangular or Cartesian coordinate system.

- **Rectangular Components of a Vector**
  - A vector $\mathbf{A}$ may have one, two or three rectangular components along the $x$, $y$ and $z$ axes, depending on orientation.
  - By two successive applications
    
    $\mathbf{A} = \mathbf{A}' + \mathbf{A}_z$
    
    $\mathbf{A}' = \mathbf{A}_x + \mathbf{A}_y$

    $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z$

    $\mathbf{A} = |\mathbf{A}|\mathbf{u}_A$
2.5 Direction Cosines of a Cartesian Vector

- Orientation of $\mathbf{A}$ is defined as the coordinate direction angles $\alpha$, $\beta$ and $\gamma$ measured between $\mathbf{A}$ and the $x$, $y$ and $z$ axes, $0^\circ \leq \alpha, \beta$ and $\gamma \leq 180^\circ$.
- The direction cosines of $\mathbf{A}$ is

$$\cos \alpha = \frac{A_x}{|\mathbf{A}|}, \quad \cos \beta = \frac{A_y}{|\mathbf{A}|}, \quad \cos \gamma = \frac{A_z}{|\mathbf{A}|}$$

$$\mathbf{u}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \left(\frac{A_x}{|\mathbf{A}|}\right)\mathbf{i} + \left(\frac{A_y}{|\mathbf{A}|}\right)\mathbf{j} + \left(\frac{A_z}{|\mathbf{A}|}\right)\mathbf{k}$$

$$\mathbf{u}_A = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\mathbf{A} = |\mathbf{A}| \mathbf{u}_A$$

$$= |\mathbf{A}| \cos \alpha \mathbf{i} + |\mathbf{A}| \cos \beta \mathbf{j} + |\mathbf{A}| \cos \gamma \mathbf{k}$$

$$= A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$$
Example 2.8

Express the force $F$ as Cartesian vector.

Since two angles are specified, the third angle is found by

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\cos^2 \alpha + \cos^2 60^\circ + \cos^2 45^\circ = 1$$

$$\cos \alpha = \pm 0.5$$

$$\alpha = \cos^{-1}(0.5) = 60^\circ \quad \text{or} \quad \alpha = \cos^{-1}(-0.5) = 120^\circ$$

By inspection, $\alpha = 60^\circ$ since $F_x$ is in the $+x$ direction

Given $F = 200\text{N}$

$$F = F \cos \alpha \mathbf{i} + F \cos \beta \mathbf{j} + F \cos \gamma \mathbf{k}$$

$$= (200 \cos 60^\circ)\mathbf{i} + (200 \cos 60^\circ)\mathbf{j} + (200 \cos 45^\circ)\mathbf{k}$$

$$= \{100.0\mathbf{i} + 100.0\mathbf{j} + 141.4\mathbf{k}\} \text{ N}$$

Checking:

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{(100.0)^2 + (100.0)^2 + (141.4)^2} = 200\text{N}$$
2.7 Position Vector: Displacement and Force

Position Displacement Vector

\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \]

\( \mathbf{F} \) can be formulated as a Cartesian vector

\[ \mathbf{F} = |\mathbf{F}| \mathbf{u} = |\mathbf{F}| \left( \frac{\mathbf{r}}{|\mathbf{r}|} \right) \]
Example 2.13

The man pulls on the cord with a force of 350N. Represent this force acting on the support A as a Cartesian vector and determine its direction.

\[ r = (3m - 0m)i + (-2m - 0m)j + (1.5m - 7.5m)k \]
\[ = \{3i - 2j - 6k\}m \]
\[ \|r\| = \sqrt{(3m)^2 + (-2m)^2 + (-6m)^2} = 7m \]

Unit vector, \( u = \frac{r}{\|r\|} = \frac{3}{7}i - \frac{2}{7}j - \frac{6}{7}k \)

Force \( F \) as a magnitude of 350N, direction specified by \( u \).

\( F = |F|u \)
\[ = 350N\left(\frac{3}{7}i - \frac{2}{7}j - \frac{6}{7}k\right) \]
\[ = \{150i - 100j - 300k\}N \]

\( \alpha = \cos^{-1}(3/7) = 64.6^\circ \)
\( \beta = \cos^{-1}(-2/7) = 107^\circ \)
\( \gamma = \cos^{-1}(-6/7) = 149^\circ \)
2.9 Dot Product

Laws of Operation

1. Commutative law
   \[ A \cdot B = B \cdot A \text{ or } A^T B = B^T A \]

2. Multiplication by a scalar
   \[ a(A \cdot B) = (aA) \cdot B = A \cdot (aB) = (A \cdot B)a \]

3. Distribution law
   \[ A \cdot (B + D) = (A \cdot B) + (A \cdot D) \]

4. Cartesian Vector Formulation
   - Dot product of 2 vectors A and B
   \[ A \cdot B = A_x B_x + A_y B_y + A_z B_z \]

5. Applications
   - The angle formed between two vectors
   \[ \theta = \cos^{-1}\left(\frac{A \cdot B}{|A||B|}\right) \quad 0^0 \leq \theta \leq 180^0 \]
   - The components of a vector parallel and perpendicular to a line
   \[ A_a = |A| \cos \theta = A \cdot u \equiv A^T u \]
Example 2.17

The frame is subjected to a horizontal force $F = \{300j\}$ N. Determine the components of this force parallel and perpendicular to the member AB.

Since
\[
\bar{u}_B = \overrightarrow{\bar{r}_B} = \frac{2i + 6j + 3k}{\sqrt{(2)^2 + (6)^2 + (3)^2}} = 0.286i + 0.857j + 0.429k
\]

Thus
\[
|\bar{F}_{AB}| = |\bar{F}| \cos \theta
\]
\[
= \bar{F} \cdot \bar{u}_B = (300j) \cdot (0.286i + 0.857j + 0.429k)
\]
\[
= (0)(0.286) + (300)(0.857) + (0)(0.429)
\]
\[
= 257.1N
\]
Solution

Since result is a positive scalar, $F_{AB}$ has the same sense of direction as $u_B$.

$$F_{AB} = |\vec{F}_{AB}|\vec{u}_{AB} = (257.1N)(0.286i + 0.857j + 0.429k)$$

$$= \{73.5i + 220j + 110k\}N$$

Perpendicular component

$$F_\perp = \vec{F} - \vec{F}_{AB} = 300j - (73.5i + 220j + 110k) = \{-73.5i + 80j - 110k\}N$$

Magnitude can be determined from $F_\perp$ or from Pythagorean Theorem

$$|F_\perp| = \sqrt{|\vec{F}|^2 - |\vec{F}_{AB}|^2} = \sqrt{(300N)^2 - (257.1N)^2} = 155N$$
Chapter 3  Equilibrium of a Particle

Chapter Objectives

• Concept of the free-body diagram for a particle
• Solve particle equilibrium problems using the equations of equilibrium

Chapter Outline

• Condition for the Equilibrium of a Particle
• The Free-Body Diagram
• Coplanar Systems
• Three-Dimensional Force Systems
3.2 The Free-Body Diagram

- **Spring**
  - Linear elastic spring: with *spring constant or stiffness* $k$. $F = k s$

- **Cables and Pulley**
  - Cables (or cords) are assumed negligible weight and cannot stretch
  - Tension always acts in the direction of the cable
  - Tension force must have a constant magnitude for equilibrium
  - For any angle, the cable is subjected to a constant tension $T$

**Procedure for Drawing a FBD**

1. Draw outlined shape
2. Show all the forces
3. Identify each of the forces
Example 3.1

The sphere has a mass of 6kg and is supported. Draw a free-body diagram of the sphere, the cord CE and the knot at C.

FBD at Sphere

Cord CE

FBD at Knot

$F_{CE}$ (Force of cord $CE$ acting on sphere)

$58.9$ N (Weight or gravity acting on sphere)

$F_{CBA}$ (Force of cord $CBA$ acting on knot)

$F_{CD}$ (Force of spring acting on knot)

$F_{CE}$ (Force of cord $CE$ acting on knot)
EXAMPLE 3.2

Determine the tension in cables BA and BC necessary to support the 60-kg cylinder in Fig. 3-6a.

SOLUTION

Equations of Equilibrium.

\[ \sum F_x = 0; \quad T_C \cos 45^\circ - \left(\frac{4}{5}\right) T_A = 0 \quad (1) \]
\[ + \sum F_y = 0; \quad T_C \sin 45^\circ + \left(\frac{3}{5}\right) T_A - 60(9.81) \text{ N} = 0 \quad (2) \]

So that

\[ T_C = 475.66 \text{ N} = 476 \text{ N} \quad T_A = 420 \text{ N} \quad \text{Ans.} \]
Example 3.7  Determine the force developed in each cable used to support the 40kN crate.

**FBD at Point A**

To expose all three unknown forces in the cables.

**Equations of Equilibrium**

Expressing each forces in Cartesian vectors,

\[
\mathbf{F}_B = F_B (\mathbf{r}_B / r_B) = -0.318 F_B \mathbf{i} - 0.424 F_B \mathbf{j} + 0.848 F_B \mathbf{k}
\]

\[
\mathbf{F}_C = F_C (\mathbf{r}_C / r_C) = -0.318 F_C \mathbf{i} - 0.424 F_C \mathbf{j} + 0.848 F_C \mathbf{k}
\]

\[
\mathbf{F}_D = F_D \mathbf{i} \quad \text{and} \quad \mathbf{W} = -40 \mathbf{k}
\]
Solution

For equilibrium,

\[ \sum \mathbf{F} = 0; \quad \mathbf{F}_B + \mathbf{F}_C + \mathbf{F}_D + \mathbf{W} = 0 \]

\[-0.318 \mathbf{F}_B \mathbf{i} - 0.424 \mathbf{F}_B \mathbf{j} + 0.848 \mathbf{F}_B \mathbf{k} - 0.318 \mathbf{F}_C \mathbf{i} \]
\[-0.424 \mathbf{F}_C \mathbf{j} + 0.848 \mathbf{F}_C \mathbf{k} + \mathbf{F}_D \mathbf{i} - 40 \mathbf{k} = 0 \]

\[\sum F_x = 0; \quad -0.318 \mathbf{F}_B - 0.318 \mathbf{F}_C + \mathbf{F}_D = 0\]
\[\sum F_y = 0; \quad -0.424 \mathbf{F}_B - 0.424 \mathbf{F}_C = 0\]
\[\sum F_z = 0; \quad 0.848 \mathbf{F}_B + 0.848 \mathbf{F}_C - 40 = 0\]

\[\rightarrow \mathbf{F}_B = \mathbf{F}_C = 23.6 \text{kN}\]
\[\mathbf{F}_D = 15.0 \text{kN}\]
Chapter 4  Force System Resultants

Chapter Objectives

• Concept of moment of a force in two and three dimensions
• Method for finding the moment of a force about a specified axis.
• Define the moment of a couple.
• Determine the resultants of non-concurrent force systems
• Reduce a simple distributed loading to a resultant force having a specified location

Chapter Outline

• Moment of a Force – Scalar Formation
• Moment of Force – Vector Formulation
• Moment of a Force about a Specified Axis
• Moment of a Couple
• Simplification of a Force and Couple System
• Reduction of a Simple Distributed Loading
4.1 Moment of a Force – Scalar Formation

• **Moment** of a force about a point or axis – a measure of the tendency of the force to cause a body to rotate about the point or axis

• Torque – tendency of rotation caused by $F_x$ or simple moment $(M_o)_z$

Magnitude

• For magnitude of $M_o$, $M_o = Fd \ (Nm)$
where $d =$ perpendicular distance from O to its line of action of force

Direction

• Direction using “right hand rule”

Resultant Moment

$M_{Ro} = \sum Fd$
4.2 Cross Product

- Cross product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) yields \( \mathbf{C} \), which is written as \( \mathbf{C} = \mathbf{A} \times \mathbf{B} = (\mathbf{A} \mathbf{B} \sin \theta)\mathbf{u}_C \)

Laws of Operations

1. Commutative law is not valid
   \[
   \mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A} \\
   \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}
   \]

2. Multiplication by a Scalar
   \[
   a(\mathbf{A} \times \mathbf{B}) = (a\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (a\mathbf{B}) = (\mathbf{A} \times \mathbf{B})a
   \]

3. Distributive Law
   \[
   \mathbf{A} \times (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})
   \]
   Proper order of the cross product must be maintained since they are not commutative.
4.2 Cross Product

Cartesian Vector Formulation

- Use \( C = \mathbf{A}\mathbf{B} \sin\theta \) on a pair of Cartesian unit vectors
- A more compact determinant in the form as

\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{vmatrix}
\]

- Moment of force \( \mathbf{F} \) about point \( O \) can be expressed using cross product

\[
\mathbf{M}_O = \mathbf{r} \times \mathbf{F}
\]

- For magnitude of cross product,

\[
\mathbf{M}_O = rF \sin\theta
\]

- Treat \( \mathbf{r} \) as a sliding vector. Since \( d = r \sin\theta \),

\[
\mathbf{M}_O = rF \sin\theta = F (r\sin\theta) = Fd
\]
4.3 Moment of Force - Vector Formulation

For force expressed in Cartesian form,

\[
M_O = r \times F = \begin{bmatrix} i & j & k \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{bmatrix}
\]

with the determinant expended,

\[
M_0 = (r_y F_z - r_z F_y)i - (r_x F_z - r_z F_x)j + (r_x F_y - r_y F_x)k
\]

\[
= \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}
\]
Determine the moment produced by the force $\mathbf{F}$ in Fig. 4–14a about point $O$. Express the result as a Cartesian vector.

**SOLUTION**

$r_A = \{12k\} \text{ m}$ and $r_B = \{4i + 12j\} \text{ m}$

$$
\mathbf{F} = F \mathbf{u}_{AB} = 2 \text{kN} \left[ \frac{\{4i + 12j - 12k\} \text{ m}}{\sqrt{(4 \text{ m})^2 + (12 \text{ m})^2 + (-12 \text{ m})^2}} \right]
$$

$$
= \begin{vmatrix}
4 & 12 & -12
\end{vmatrix}
\begin{bmatrix}
\frac{r_{AB}}{||r_{AB}||}
\end{bmatrix}
= \{0.4588i + 1.376j - 1.376k\} \text{ kN}
$$

Thus

$$
\mathbf{M}_O = r_A \times \mathbf{F} = \begin{vmatrix}
i & j & k \\
0 & 0 & 12 \\
0.4588 & 1.376 & -1.376
\end{vmatrix}
$$

$$
= \mathbf{M}_O = r_B \times \mathbf{F} = \begin{vmatrix}
i & j & k \\
4 & 12 & 0 \\
0.4588 & 1.376 & -1.376
\end{vmatrix}
$$

or $$
= \frac{r_{AB}}{||r_{AB}||} \begin{bmatrix}
0 & 12 & 0
\end{bmatrix} \begin{bmatrix}
4
\end{bmatrix}
= \begin{bmatrix}
0 & 12 & 0 \\
-12 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
12
\end{bmatrix}
= \{-16.5i + 5.51j\} \text{ kN\cdot m}
$$

*Ans.*
Example 4.4

Two forces act on the rod. Determine the resultant moment they create about the flange at \( O \). Express the result as a Cartesian vector.

\[
\mathbf{r}_A = \{5j\} \text{ m}
\]

\[
\mathbf{r}_B = \{4i + 5j - 2k\} \text{ m}
\]

The resultant moment about \( O \) is

\[
\mathbf{M}_O = \sum (\mathbf{r} \times \mathbf{F}) = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times \mathbf{F}
\]

\[
= \begin{bmatrix} i & j & k \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{bmatrix} \begin{bmatrix} i & j & k \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & 0 & -5 \\ 0 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} -60 \\ 40 \\ 20 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & 4 \\ 5 & -4 & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 40 \\ -30 \end{bmatrix}
\]

\[
= \{30i - 40j + 60k\} \text{ kN} \cdot \text{m}
\]
4.4 Principles of Moments

- Since $F = F_1 + F_2$,

$$M_O = r \times F = r \times (F_1 + F_2) = r \times F_1 + r \times F_2$$

4.5 Moment of a Force about a Specified Axis Vector Analysis

- For magnitude of $M_A$,

$$M_A = M_O \cos \theta = M_O \cdot u_a$$

where $u_a = \text{unit vector}$

- In determinant form,

$$\left| M_a \right| = u_{ax} \cdot (r \times F) = \begin{vmatrix} u_{ax} & u_{ay} & u_{az} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$
Example 4.8  Determine the moment produced by the force $F$ which tends to rotate the rod about the $AB$ axis.

\[ r_c = \begin{bmatrix} 0.6 \\ 0 \\ 0.3 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 0 \\ -300 \end{bmatrix} \]

\[ M = r \times F = \begin{bmatrix} 0 & -0.3 & 0 \\ 0.3 & 0 & -0.6 \\ 0 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -300 \end{bmatrix} = \begin{bmatrix} 0 \\ 180 \\ 0 \end{bmatrix} \]

\[ r_B = \begin{bmatrix} 0.4 \\ 0.2 \\ 0 \end{bmatrix}, \quad u_B = \begin{bmatrix} 0.4 \\ 0.2 \\ 0 \end{bmatrix} \frac{1}{\sqrt{0.2}} \]

\[ M_{AB} = u_B^T M = \begin{bmatrix} 0.4 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 180 \\ 0 \end{bmatrix} \frac{1}{\sqrt{0.2}} = 80.4 \]
Determine the magnitude of the moment of force \( \mathbf{F} \) about segment \( OA \) of the pipe assembly in Fig. 4–24a.

\[
\mathbf{r}_{CD} = \begin{bmatrix} 0.4 & -0.4 & 0.2 \end{bmatrix}^T
\]

\[
\mathbf{F} = \frac{300 \mathbf{r}_{CD}}{\| \mathbf{r}_{CD} \|}
\]

\[
\mathbf{r}_{OC} = [0.1, 0.4, 0.3]
\]

\[
\mathbf{M} = \mathbf{r}_{OC} \times \mathbf{F} = \begin{bmatrix} 0 & -0.3 & 0.4 \\ 0.3 & 0 & -0.1 \\ -0.4 & 0.1 & 0 \end{bmatrix} \begin{bmatrix} 200 \\ -200 \\ 100 \end{bmatrix} = \begin{bmatrix} 100 \\ 50 \\ -100 \end{bmatrix}
\]

\[
\mathbf{M}_{OA} = \mathbf{r}_{OA}^T \mathbf{M} \left( \frac{1}{\| \mathbf{r}_{OA} \|} \right) = \begin{bmatrix} 0.6 & 0.8 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 50 \\ -100 \end{bmatrix} = 100
\]
4.6 Moment of a Couple

- Couple - two parallel forces of the same magnitude but opposite direction separated by perpendicular distance $d$

Scalar Formulation

- Magnitude of couple moment $M = Fd$
- $M$ acts perpendicular to plane containing the forces

Vector Formulation

- For couple moment, $M = r \times F$

Equivalent Couples

- 2 couples are equivalent if they produce the same moment
- Forces of equal couples lie on the same plane or plane parallel to one another
Example 4.12

Determine the couple moment acting on the pipe. Segment $AB$ is directed $30^\circ$ below the $x$–$y$ plane.

Take moment about point O,

$$\mathbf{M} = \mathbf{r}_A \times (-250\mathbf{k}) + \mathbf{r}_B \times (250\mathbf{k})$$

$$= (0.8\mathbf{j}) \times (-250\mathbf{k}) + (0.66\cos30^\circ\mathbf{i} + 0.8\mathbf{j} - 0.6\sin30^\circ\mathbf{k}) \times (250\mathbf{k})$$

$$= \{-130\mathbf{j}\} \text{N.cm}$$

Take moment about point A

$$\mathbf{M} = \mathbf{r}_{AB} \times (250\mathbf{k})$$

$$= (0.6\cos30^\circ\mathbf{i} - 0.6\sin30^\circ\mathbf{k}) \times (250\mathbf{k})$$

$$= \{-130\mathbf{j}\} \text{N.cm}$$

Take moment about point A or B,

$$\mathbf{M} = Fd = 250\text{N}(0.5196\text{m}) = 129.9\text{N.cm}$$

$\mathbf{M}$ acts in the $-\mathbf{j}$ direction $\mathbf{M} = \{-130\mathbf{j}\} \text{N.cm}$
4.7 Simplification of a Force and Couple System

- Equivalent resultant force acting at point $O$ and a resultant couple moment is expressed as

$$F_R = \sum F$$

$$\left( M_R \right)_O = \sum M_O + \sum M$$

- If force system lies in the $x$–$y$ plane, then the couple moments are perpendicular to this plane,

$$\left( F_R \right)_x = \sum F_x$$

$$\left( F_R \right)_y = \sum F_y$$

$$\left( M_R \right)_O = \sum M_O + \sum M$$
4.8 Simplification of a Force and Couple System

Concurrent Force System

- A *concurrent force system* is where lines of action of all the forces intersect at a common point $O$

\[
R = \sum F
\]

Coplanar Force System

- Lines of action of all the forces lie in the same plane
- Resultant force of this system also lies in this plane
EXAMPLE 4.19

The slab in Fig. 4–46a is subjected to four parallel forces. Determine the magnitude and direction of a resultant force equivalent to the given force system and locate its point of application on the slab.

**Fig. 4–46**

**SOLUTION (SCALAR ANALYSIS)**

**Force Summation.** From Fig. 4–46a, the resultant force is

\[ + \mathbf{F}_R = \Sigma \mathbf{F}; \quad - \mathbf{F}_R = -600 \text{ N} + 100 \text{ N} - 400 \text{ N} - 500 \text{ N} \]

\[ = -1400 \text{ N} = 1400 \text{ N} \downarrow \quad \text{Ans.} \]

**Moment Summation.**

\[ (M_R)_x = \Sigma M_x; \]

\[ -(1400 \text{ N}) y = 600 \text{ N}(0) + 100 \text{ N}(5 \text{ m}) - 400 \text{ N}(10 \text{ m}) + 500 \text{ N}(0) \]

\[ y = 2.50 \text{ m} \quad \text{Ans.} \]

In a similar manner

\[ (M_R)_y = \Sigma M_y; \]

\[ (1400 \text{ N}) x = 600 \text{ N}(8 \text{ m}) - 100 \text{ N}(6 \text{ m}) + 400 \text{ N}(0) + 500 \text{ N}(0) \]

\[ x = 3 \text{ m} \quad \text{Ans.} \]
4.9 Distributed Loading

- Large surface area of a body may be subjected to distributed loadings, often defined as pressure measured in Pascal (Pa): \(1 \text{ Pa} = 1 \text{N/m}^2\)

Magnitude of Resultant Force
- Magnitude of \(dF\) is determined from differential area \(dA\) under the loading curve.
- For length \(L\), \(F_R = \int w(x)dx = \int dA = A\)

Location of Resultant Force
- \(dF\) produces a moment of \(xdF = x \cdot w(x) \cdot dx\) about \(O\)
- \(\bar{x}F_R = \int xw(x)dx\)
- Solving for \(\bar{x}\)
  \[
  \bar{x} = \frac{\int xw(x)dx}{\int w(x)dx}
  \]
Example 4.21

Determine the magnitude and location of the equivalent resultant force acting on the shaft.

For the differential area element,
\[ dA = wxdx = 60x^2dx \]

For resultant force
\[ F_R = \int_0^2 dA = \int_0^2 60x^2dx \]
\[ = 60 \left[ \frac{x^3}{3} \right]_0^2 = 60 \left[ \frac{2^3}{3} - 0 \right] = 160N \]

For location of line of action,
\[ \bar{x} = \frac{\int_0^2 x(60x^2)dx}{\int_0^2 dA} = \frac{60 \left[ \frac{x^4}{4} \right]_0^2}{160} = \frac{60 \left[ \frac{2^4}{4} - \frac{0^4}{4} \right]}{160} = 1.5m \]
Chapter 5 Equilibrium of a Rigid Body

Objectives

• Equations of equilibrium for a rigid body
• Concept of the free-body diagram for a rigid body

Outline

• Conditions for Rigid Body Equilibrium
• Free-Body Diagrams
• Two and Three-Force Members
• Equations of Equilibrium
• Constraints and Statical Determinacy
5.1 Conditions for Rigid-Body Equilibrium

- The equilibrium of a body is expressed as

\[ F_R = \sum F = 0 \]

\[ (M_R)_O = \sum M_O = 0 \]

- Consider summing moments about some other point, such as point A, we require

\[ \sum M_A = r \times F_R + (M_R)_O = 0 \]
5.2 Free Body Diagrams

- If a support prevents the translation of a body in a given direction, then a force is developed on the body in that direction.
- If rotation is prevented, a couple moment is exerted on the body.
### 5.2 Free Body Diagrams

<table>
<thead>
<tr>
<th>Types of Connection</th>
<th>Reaction</th>
<th>Number of Unknowns</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Cable</td>
<td><img src="image1" alt="Cable Diagram" /></td>
<td>One unknown. The reaction is a tension force which acts away from the member in the direction of the cable</td>
</tr>
<tr>
<td>(2) Weightless Link</td>
<td><img src="image2" alt="Weightless Link Diagram" /></td>
<td>One unknown. The reaction is a force which acts along the axis of the link.</td>
</tr>
<tr>
<td>(3) Roller</td>
<td><img src="image3" alt="Roller Diagram" /></td>
<td>One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.</td>
</tr>
<tr>
<td>(4) Roller or Pin in Confined Smooth Slot</td>
<td><img src="image4" alt="Roller or Pin Diagram" /></td>
<td>One unknown. The reaction is a force which acts perpendicular to the slot.</td>
</tr>
<tr>
<td>(5) Rocker</td>
<td><img src="image5" alt="Rocker Diagram" /></td>
<td>One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.</td>
</tr>
</tbody>
</table>
### 5.2 Free Body Diagrams

<table>
<thead>
<tr>
<th>Diagram</th>
<th>Description</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Diagram 1" /></td>
<td>smooth contacting surface</td>
<td>One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.</td>
</tr>
<tr>
<td><img src="image2.png" alt="Diagram 2" /></td>
<td>member pin connected</td>
<td>One unknown. The reaction is a force which acts perpendicular to the rod.</td>
</tr>
<tr>
<td><img src="image3.png" alt="Diagram 3" /></td>
<td>smooth pin or hinge</td>
<td>Two unknowns. The reactions are two components of force, or the magnitude and direction $\phi$ of the resultant force. Note that $\phi$ and $\theta$ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].</td>
</tr>
<tr>
<td><img src="image4.png" alt="Diagram 4" /></td>
<td>Member fixed connected to collar on smooth rod</td>
<td>Two unknowns. The reaction are the couple moment and the force which acts perpendicular to the rod.</td>
</tr>
<tr>
<td><img src="image5.png" alt="Diagram 5" /></td>
<td>fixed support</td>
<td>Three unknowns. The reaction are the couple moment and the two force components, or the couple moment and the magnitude and direction $\phi$ of the resultant force.</td>
</tr>
</tbody>
</table>
5.2 Free Body Diagram

Weight and Center of Gravity

- Each particle has a specified weight
- System can be represented by a single resultant force, known as weight $W$ of the body
- Location of the force application is known as the center of gravity
Example 5.1  Draw the free-body diagram of the uniform beam. The beam has a mass of 100kg.

Free-Body Diagram
• Support at A is a fixed wall
• Two forces acting on the beam at A denoted as $A_x, A_y$, with moment $M_A$
• For uniform beam,
  Weight, $W = 100(9.81) = 981N$
  acting through beam’s center of gravity
5.4 Two- and Three-Force Members

- When forces are applied at only two points on a member, the member is called a two-force member.
- Only force magnitude must be determined.

Three-Force Members
When subjected to three forces, the forces are concurrent or parallel.
### 5.5 3D Free-Body Diagrams

<table>
<thead>
<tr>
<th>Types of Connection</th>
<th>Reaction</th>
<th>Number of Unknowns</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) cable</td>
<td><img src="cable.png" alt="Image" /></td>
<td>One unknown. The reaction is a force which acts away from the member in the known direction of the cable.</td>
</tr>
<tr>
<td>(2) smooth surface support</td>
<td><img src="smooth_surface.png" alt="Image" /></td>
<td>One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.</td>
</tr>
<tr>
<td>(3) roller</td>
<td><img src="roller.png" alt="Image" /></td>
<td>One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.</td>
</tr>
<tr>
<td>(4) ball and socked</td>
<td><img src="ball_and_socked.png" alt="Image" /></td>
<td>Three unknown. The reaction are three rectangular force components.</td>
</tr>
<tr>
<td>(5) single journal bearing</td>
<td><img src="single_journal.png" alt="Image" /></td>
<td>Four unknown. The reaction are two force and two couple moment components which acts perpendicular to the shaft. Note: The couple moments are generally not applied of the body is supported elsewhere. See the example.</td>
</tr>
</tbody>
</table>
5.7 Constraints for a Rigid Body

Redundant Constraints

- More support than needed for equilibrium
- Statically indeterminate: more unknown loadings than equations of equilibrium
5.7 Constraints for a Rigid Body

Improper Constraints

- Instability caused by the improper constraining by the supports
- When all reactive forces are concurrent at this point, the body is improperly constrained