Airplane Performance Analysis

In general, performance analysis typically includes the following phases: take-off, climbing, cruise, descent, approach, landing and accelerated performance. The airplane is assumed to be trimmed (i.e. zero moments), so the whole airplane is assumed to be a point mass. The needed aerodynamic data are $C_L-C_D$ curves at various Mach numbers, and at different flap angles plus gears. Typical curves are presented in the following figures.

Cruise Performance

Because the cruise performance is very important in transport airplane design, its analysis for a jet airplane only will be illustrated here. The jet engine performance is defined in terms of fuel flow rate in lbs/hr in producing one pound of thrust, called the specific fuel consumption ($c_j$). If $T_{req}$ is the required thrust in cruise, then the actual fuel consumption rate would be:

$$\dot{W}_F = T_{req} c_j$$

(1)

The weight reduction is equal to the fuel consumption. Therefore,
\[ dW = -\dot{W}_F dt = -T_{req} c_j dt \]  \hspace{1cm} (2)

The specific endurance, or endurance factor, is defined as the time in hrs flown per pound of fuel and is given by:

\[ \frac{dt}{dw} = -\frac{1}{T_{req} c_j}, \text{hrs/lbs} \]  \hspace{1cm} (3)

The specific range, or range factor, is defined as the distance travelled per pound of fuel and is given by: (using the relation: \( V = ds/dt \))

\[ \frac{ds}{dW} = -\frac{V}{T_{req} c_j}, \text{nm/lbs} \]  \hspace{1cm} (4)

Since

\[ T_{req} = D = \frac{D}{L} W = \frac{C_D}{C_L} W \]

it follows that

\[ \frac{ds}{dW} = -\frac{V (C_L / C_D)}{W c_j} \]  \hspace{1cm} (5)

But the flight speed can be written as:

\[ V = \sqrt{ \frac{2W}{\rho S C_L} } \]  \hspace{1cm} (6)

Therefore,

\[ \frac{ds}{dW} = -\frac{1}{1.689 c_j \sqrt{\rho S C_D}} \left[ \frac{\sqrt{C_L}}{C_D} \right] dW, \text{ in nm/lbs} \]  \hspace{1cm} (7)

Eq. (7) shows that for a given weight and altitude, the range is maximum if \( \sqrt{C_L / C_D} \) is maximized.

Assuming constant air density, \( c_j \) and \( \sqrt{C_L / C_D} \), Eq. (7) can be integrated exactly to give:

\[ R = \frac{1.675}{c_j \sqrt{\rho S C_D}} \sqrt{\frac{C_L}{C_D}} \left( \sqrt{W_{begin}} - \sqrt{W_{end}} \right), \text{ in nm} \]  \hspace{1cm} (8)

For the endurance, Eq. (3) is written as

\[ \frac{dt}{dW} = -\frac{C_L / C_D}{W c_j} \]

Therefore, the endurance is given by:
Eqs. (8) and (9) are called Breguet equations for range and endurance of jet airplanes.

On the other hand, a jet airplane may cruise at a constant true speed or Mach number. In this case, Eq. (5) is directly integrated to give, with constant \( c_j \) and \( C_L/C_D \),

\[
R = \frac{V C_L / C_D}{c_j} \ln \left( \frac{W_{\text{begin}}}{W_{\text{end}}} \right)
\]

Since \( V = M V_a \), where \( V_a \) is the speed of sound, Eq. (10) can also be written as:

\[
R = \left[ \frac{V_a M C_L / C_D}{c_j} \right] \ln \left( \frac{W_{\text{begin}}}{W_{\text{end}}} \right)
\]

Eq. (10) is frequently used to determine the combat radius of a fighter at a specified speed. Eq. (11) is further used to determine the best cruise Mach number as shown in the figure below.

The constant speed endurance is still given by Eq. (9).

As shown in Eq. (8), to maximize the range, the ratio, \( \sqrt{C_L / C_D} \) must be maximized. In cruise, a parabolic drag equation is a valid assumption. Therefore, the following ratio is to be maximized:

\[
f = \frac{\sqrt{C_L}}{C_{D0} + kC_L^2}
\]
where \( k = 1/\pi A_e \). By setting \( df/dC_L \) to 0, it can be shown that

\[
C_{D0} = 3kC_L^2
\]  

(12)

On the other hand, if \( C_L/C_D \) is to be maximized, then

\[
C_{D0} = kC_L^2
\]  

(13)

Eq. (12) is used to determine \( C_L \) for the maximum range and hence the speed through Eq. (6). Based on Eq. (12), \( C_L \) obtained tends to be small, and hence the flight speed is high. If the corresponding Mach number exceeds the drag divergent Mach number, the flight speed must be reduced to one corresponding to a Mach number slightly below \( M_{div} \).

**Example:** A small jet transport airplane has the following characteristics:

\[
\begin{align*}
W_{TO} & = 50,000 \text{ lbs} & W_{OWE} & = 28,000 \text{ lbs} \\
C_D & = 0.0190 + 0.055 C_L^2 & h & = 35,000 \text{ ft} \\
S & = 500 \text{ ft}^2 & c_j & = 0.65 \text{ lbs/hr/lbs at } M = 0.75 \text{ and } h = 35,000 \text{ ft} \\
A & = 8
\end{align*}
\]

Assume that the begin weight in the cruise phase is: \( W_{CR_{beg}} = 49,000 \text{ lbs} \). Assume that the end weight in the cruise phase is: \( W_{CR_{end}} = 39,000 \text{ lbs} \). Determine the cruise range and the cruise endurance at constant Mach number.

**Solution:**

The calculation will be performed by using Eqs (9) and (11).

For range, the following input data are available for Eqn (11):

\[
V_{a_0} = 661.5 \text{ kts} \\
\sqrt{\theta} = 0.8714 \text{ at 35,000 ft}
\]

In the cruise phase at \( M=0.75 \) at 35,000 ft, where: \( q/M^2 = 348.6 \text{ psf} \), the dynamic pressure may be found from: \( q = (0.75)^2 348.6 = 196.1 \text{ psf} \).

Since \( L/D \) will vary during the mission, an average will be used. The following lift coefficients are found for the beginning and end of cruise:

\[
\begin{align*}
C_{L_{CR_{beg}}} & = 49,000/196.1 \times 555 = 0.50 \\
C_{L_{CR_{end}}} & = 39,000/196.1 \times 555 = 0.40
\end{align*}
\]
With the cruise drag polar, this yields the following values for L/D:

\[
(C_L/C_D)_{\text{begin}} = 0.50/0.0328 = 15.2
\]

\[
(C_L/C_D)_{\text{end}} = 0.40/0.0278 = 14.4
\]

The average L/D value of the airplane in cruise is 14.8. The range now follows from Eqn (\ref{eqn:15}) as:

\[
R = \left(\frac{661.5 \times 0.8714 \times 0.75 \times 14.8}{0.65}\right) \ln \frac{49,000}{39,000} = 2,247 \text{ nm}
\]

The endurance in this cruise condition follows from Eqn (\ref{eqn:9}) as:

\[
E = \left(\frac{14.8}{0.65}\right) \ln \frac{49,000}{39,000} = 5.2 \text{ hrs}
\]

This finding should be checked with the flight speed of \(V = 0.75 \times 576.4 = 432.3 \text{ kts.}\) Diving this into 2,247 yields for the flight time: 5.2 hrs!

It is noted with Eqn (\ref{eqn:13}) \(\frac{C_L}{C_D}_{\text{max}} = 15.5\) at \(C_L = 0.59\).

**Example 2:** For the same jet transport airplane of Example 1, determine the maximum range for constant altitude and for constant Mach number. Discuss the results.

**Solution:** The calculation will be performed by using Eqn (\ref{eqn:11}).

Eqn (\ref{eqn:7}) will be used for the calculation of maximum range at constant altitude. The density of air at 35,000 ft is: \(\rho = 0.0007369 \text{ slugs/ft}^3\).

From Eqn (\ref{eqn:12}) the lift and drag coefficients for maximum \(\sqrt{C_L} / C_D\) are respectively:

\[
C_{L(\sqrt{C_L}/C_D)_{\text{max}}} = \sqrt{\frac{\pi AeC_D}{3}} = 0.34 \quad C_{D(\sqrt{C_L}/C_D)_{\text{max}}} = \frac{4}{3} C_D = 0.0253
\]

The maximum value of \(\sqrt{C_L} / C_D\) now is: \(0.34 / 0.0253 = 23.0\). This yields for the best range at constant altitude:

\[
R = \frac{1.675}{0.65 \times \sqrt{0.0007369 \times 500}} \times 23.0 \times \left(\sqrt{49,000} - \sqrt{39,000}\right) = 2,331 \text{ nm}
\]
By using $L/D_{max} = 15.5$ at $C_L = 0.59$ in Eqn ($\frac{L}{D}$) it follows that:

$R = 2,353$ nm.

Discussion: this shows that flying at constant Mach number is marginally better than flying at constant altitude. It is of interest to check the speed for the constant altitude case. Eqn ($\frac{L}{D}$) yields for the speed at the start of cruise:

$$V = \sqrt{\frac{W}{\rho CL_{max}} S} = \sqrt{\frac{49,000}{500 \cdot 0.0007369 \cdot 0.34}} = 884.5 \text{ ft/sec} = 524 \text{ kts}$$

This speed represents a Mach number of 0.91 which is way too high for such a transport. The reason for this absurd result is that the drag polar was kept constant with Mach number. For an airplane designed for a low cruise Mach number ($M=0.75$), the drag rise would be considerable.

Stall Speeds and Minimum Speeds

The 1-g stall speed is defined by

$$V_s = \sqrt{\frac{2W}{\mu SC_{L_{max,trim}}}}$$ (14)

However, if the airplane is not controllable at $\alpha_{stall}$, $C_{L_{max}}$ in Eq. (14) must be replaced with $C_{L_{max, controllable}}$. Because $V_s$ affects greatly the performance of the airplane in take-off, landing, approach and climb, it is required to determine it through flight test with the requirements that the speed reduction to the minimum speed does not exceed one knot per second; the C.G. at the most critical location (usually means the most forward); and at zero thrust. Of course, it is also evaluated at various flap angles.

Level Flight Maximum Speeds and Ceilings

Maximum thrust is a function of altitude and Mach number: $T_{max}(\delta,M)$, where $\delta=p/p_0$. With a parabolic drag equation, $T_{max}(\delta,M)$ can be written as:

$$\frac{T_{max}(\delta,M)}{W/\delta} = C_{D0}(M)\bar{q}S + \frac{W/\delta}{\bar{q}S\pi\alpha e}$$

$$\bar{q} = 1481.3\delta M^2$$ (15)

Eq. (15) can be used to graphically determine the maximum level flight Mach numbers and absolute ceilings at a given weight and altitude.

Flight Envelope

A typical V-n diagram is presented in the following.
Note that the stall speed in the V-n diagram may be computed from

\[ V_s = \sqrt{\frac{2W_d}{\rho C_{N_{\text{max}}} S}} \quad , \quad C_{N_{\text{max}}} = \sqrt{(C_{L_{\text{max}}}^2 + C_D^2)} \quad , \quad C_D \text{ at } C_{L_{\text{max}}} \]  \hspace{1cm} (16)

\( V_A \) is the design maneuvering speed and is estimated as

\[ V_A = V_s \sqrt{n_{\text{lim}}} \]

where \( n_{\text{lim}} = L_{\text{lim}}/W \) (the load factor) and is specified as 2.5 for a jet transport.

\( V_C \) is the design cruising speed and is defined by the designer.

\( V_D \) is the design diving speed and must satisfy:

\[ V_D \geq 1.25V_C \]

**Maneuvering in a vertical plane**

In an instantaneous pull-up maneuver, the lift is equal to \( nW \). Therefore, from an aerodynamic point of view,
It is seen from Eq. (17) that when $n$ is plotted versus speed, the curve is parabolic (i.e. varies with $V^2$). However, in a sustained pull-up maneuver, there must be enough thrust to overcome the drag. In this case, “$n$” is solved from:

$$T = D = (C_{D0} + \left(\frac{nC_L}{\pi A_e}\right)^2)\frac{\rho}{\pi} S$$ \hspace{1cm} (18)

At a high speed, a jet airplane is mostly limited by the available thrust in maneuvering.

**Maneuvering in a horizontal plane**

![Diagram of a steady, level turn](image)

In a steady level and coordinated turn, the following equations are satisfied:

$$L \cos \phi = W$$ \hspace{1cm} (19)
\[ L \sin \phi = \frac{W \ V^2}{g \ R_t} = C.F. \]  \hfill (20)

Note that “coordinated turn” means there is no side force. From Eq. (19), the load factor in a level and coordinated turn is therefore

\[ n = \frac{L}{W} = \frac{1}{\cos \phi} \]  \hfill (21)

The radius of turn is obtained from Eq. (20):

\[ R_t = \frac{V^2 \ g \ \tan \phi}{g \sqrt{n^2 - 1}} \]  \hfill (22)

The turn rate is given by

\[ \psi = \frac{V}{R_t} \]  \hfill (23)

Note that in any maneuver, the stall speed becomes a function of the load factor. Eq. (17) is still valid:

\[ V_{s(\text{turn})} = V_s(1-g) \sqrt{n} \]  \hfill (24)

Example 3 An airplane is flying straight and level at sea-level and at a speed of 300 ft/sec. The pilot puts the airplane in a level, coordinated turn with a radius of 2,850 ft, while maintaining the same angle of attack as the one the airplane had in the straight and level flight condition. The pilot adjusts the engine thrust as required to maintain the speed at 300 ft/sec (i.e. sustained turn). Determine the required thrust.
Solution:

In straight and level flight: \[ L_{\text{level}} = W \]

In level, coordinated turning flight: \[ L_{\text{turn}} \cos \phi = W = L_{\text{level}} \]

As long as the angle of attack remains the same, the lift-to-drag ratio will remain the same. Therefore:

\[ (L/D)_{\text{turn}} = (L/D)_{\text{level}} \]
\[ V_{\text{turn}}^2 \cos \phi = V_{\text{level}}^2 \]

In a steady level turn, according to Eqn (27.2):

\[ \tan \phi = \frac{V_{\text{turn}}^2}{gR_t} = \frac{V_{\text{level}}^2}{gR_t \cos \phi} \]

Therefore:

\[ \sin \phi = \frac{V_{\text{level}}^2}{gR_t} = \frac{300^2}{32.2 \times 2,850} = 0.9807 \]

The ratio of the thrust required to overcome the drag in the turn to that in level flight is found from:

\[ \frac{T_{\text{turn}}}{T_{\text{level}}} = \frac{D_{\text{turn}}}{D_{\text{level}}} = \frac{L_{\text{turn}}}{L_{\text{level}}} \cos \phi = \frac{1}{\sqrt{1 - 0.9807^2}} = 5.12 \]