1. Write down the complex potential for a source of strength $m$ located at $z = i\alpha$ and a source of strength $m$ located at $z = -i\alpha$. Show that the real axis is a streamline in the resulting flow field, and so deduce that the complex potential for the two sources is also the complex potential for a flat plate located along $y = 0$ with a source of strength $m$ located a distance $h$ above it.

Obtain the pressure on the surface of the plate mentioned above from the Bernoulli equation. Integrate this pressure over the entire surface of the plate, and so show that the force acting on the plate, due to the presence of the source, is $\rho m^2/(4\pi h)$. Take the pressure below the plate to be equal to the stagnation pressure in the fluid.

(25%)

2. Obtain the complex potential for a source of strength $m$ located at $z = be^{i(\alpha+\pi)}$, a source of strength $m$ located at $z = (a^2/b)e^{i(\alpha+\pi)}$, a sink of strength $m$ located at $z = (a^2/b)e^{i\alpha}$, a sink of strength $m$ located at $z = be^{i\alpha}$, and a constant term of magnitude $-im/2$. Expand this result for small values of $z/b$ and $a^2/(bz)$, and hence show that as $b \to \infty$ and as $m \to \infty$ such that $m/b \to \pi U$, the circle of radius $a$ is a streamline. Hence show that the complex potential for a uniform flow of magnitude $U$ approaching a circular cylinder of radius $a$ at an angle of attack $\alpha$ to the horizontal is

$$F(z) = U \left( ze^{-i\alpha} + \frac{a^2}{z} e^{i\alpha} \right)$$

(4.29)

(25%)

3. Determine the complex potential for a circular cylinder of radius $a$ in a flow field which is produced by a counterclockwise vortex of strength $\Gamma$ located a distance $l$ from the axis of the cylinder. This may be done by writing the complex potential for a clockwise vortex of strength $\Gamma$ located at $z = a^2/l$, a counterclockwise vortex of strength $\Gamma$ located at $z = l$, and a constant of magnitude $-(i\Gamma/(2\pi)) \log b$. Then let $b \to \infty$ and show that the circle of radius $a$ is a streamline.

Obtain the value of the force acting on the cylinder by applying the Blasius law to a contour which includes the cylinder but excludes the vortex at $z = l$.

(25%)
4.

Use the Schwarz-Christoffel transformation to find the mapping which transforms the interior of the 90° bend shown in the $z$ plane of Fig. 4.24 onto the upper half of the $\zeta$ plane as shown. Hence obtain the complex potential for the flow around in a right-angled bend in terms of the channel width $l$ and the approach velocity $U$.

**Figure 4.24**
Mapping planes for flow in a channel having a 90° bend.

(25%)
\[ F(z) = \frac{m}{2\pi \lambda} \ln (z - \lambda h) + \frac{m}{2\pi \lambda} \ln (z + \lambda h) \rightarrow \text{Complex potential} \]

\[ F(z) = \frac{m}{2\pi \lambda} \ln ((z - \lambda h)(z + \lambda h)) = \frac{m}{2\pi \lambda} \ln (z^2 + \lambda^2) \]

In real axis \( \Rightarrow z = a \in \mathbb{R} \Rightarrow F(z) = \frac{m}{2\pi \lambda} \ln (a^2 + \lambda^2) \in \mathbb{R} \)

\[ \psi = \phi + i\psi_i \Rightarrow \phi = \frac{m}{2\pi \lambda} \ln (a^2 + \lambda^2) \quad \psi_i = 0 \]

\[ \psi = 0 = \text{constant is streamline} \]

\[ \frac{dy}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi_i}{\partial y} = -\nu \frac{d\phi}{dx} \Rightarrow \text{streamline} \]

\[ W(z) = \frac{dF(z)}{dz} = \frac{m}{2\pi \lambda} \left[ \frac{1}{z - \lambda h} + \frac{1}{z + \lambda h} \right] = \frac{m}{2\pi \lambda} \left( \frac{2z}{z^2 + \lambda^2} \right) \]

By Bernoulli's equation

\[ P_s + \frac{1}{2} \rho v_s^2 = P_0 + \frac{1}{2} \rho v_0^2 \]

\[ P_s = P_0 - \frac{1}{2} \rho v_s^2 \]

\[ \rho = P_0 - \frac{1}{2} \rho \frac{m^2}{4\pi \lambda^2} \left[ \frac{1}{z - \lambda h} + \frac{1}{z + \lambda h} \right] \]

Res \( W(z) \) | \( z = \lambda h \) = \( \frac{1}{\lambda h} \frac{m^2}{4\pi \lambda^2} \)

\[ F(z) = -\int_C (P_s - P_0) \, dz = \frac{1}{2} \rho \frac{m^2}{4\pi \lambda^2} \int_C (\frac{1}{z - \lambda h} + \frac{1}{z + \lambda h}) \, dz \]

\[ = \frac{1}{2} \rho \frac{m^2}{4\pi \lambda^2} \lambda h (2\lambda h) \]

\[ = \frac{\rho m^2}{4\pi \lambda h} \]
\[ F(z) = \frac{m}{2\pi} \ln \left[ z - be^{i(\alpha + \lambda)} \right] + \frac{m}{2\pi} \ln \left[ z - \frac{a^2}{b} e^{i(\alpha + \lambda)} \right] - \frac{m}{2\pi} \ln \left[ z - \frac{a^2}{b} e^{i\lambda} \right] - \frac{\lambda m}{2} \]

\[ = \frac{m}{2\pi} \ln \left[ z - b(-1) e^{i\lambda} \right] + \frac{m}{2\pi} \ln \left[ z - \frac{a^2}{b} (-1) e^{i\lambda} \right] - \frac{m}{2\pi} \ln \left[ z - b e^{i\lambda} \right] - \frac{\lambda m}{2} \quad (e^{i\lambda} = -1) \]

\[ = \frac{m}{2\pi} \ln \left( \frac{z + b e^{i\lambda}}{z - b e^{i\lambda}} \right) + \frac{m}{2\pi} \ln \left( \frac{z + \frac{a^2}{b} e^{i\lambda}}{z - \frac{a^2}{b} e^{i\lambda}} \right) - \frac{\lambda m}{2} \]

\[ = \frac{m}{2\pi} \ln \left( \frac{z^2 + b^2 e^{2i\lambda}}{z^2 - b^2 e^{2i\lambda}} \right) + \frac{m}{2\pi} \ln \left( \frac{1 + \frac{a^2}{b} e^{i\lambda}}{1 - \frac{a^2}{b} e^{i\lambda}} \right) - \frac{\lambda m}{2} \]

\[ = \frac{m}{2\pi} \left( 2 \cdot \frac{2}{b} e^{-i\lambda} \right) + \frac{m}{2\pi} \left( 2 \cdot \frac{a^2}{b^2} e^{i\lambda} \right) - \frac{\lambda m}{2} + \frac{\lambda m}{2} \]

\[ = \frac{m}{2\pi} \frac{2}{b} e^{-i\lambda} + \frac{m}{2\pi} \frac{a^2}{b^2} e^{i\lambda} \quad \text{with } b \to \infty, \ m \to \infty, \ \frac{m}{b} \to \nu \pi \]

\[ F(z) = \nu z e^{-i\lambda} + \frac{a^2}{2} \nu e^{i\lambda} = \nu (ze^{-i\lambda} + \frac{a^2}{2} e^{i\lambda}) \]

\[ z = a \text{ should be a streamline } \Rightarrow z = ae^{i\theta} \]

\[ vz e^{-i\lambda} + \frac{a^2}{a e^{i\lambda}} \nu e^{i\lambda} \]

\[ = vz \left( e^{-i\lambda} + e^{i(\theta - \lambda)} \right) \]

\[ = vz \left( e^{i(\theta - \lambda)} + e^{-i(\theta - \lambda)} \right) \Rightarrow e^{ix} + e^{-ix} = 2 \cos x \]

\[ = 2vz \cos (\theta - \lambda) = \phi + i \cdot 4^0 \]

\[ \psi = 0 \Rightarrow \text{the circle of radius } a \text{ is a streamline.} \]
3. \[ F(z) = \frac{i\nu}{2\pi} \ln \left( z - \frac{\alpha^2}{\lambda} \right) - \frac{i\nu}{2\pi} \ln \left( z - \frac{1}{\lambda} \right) - \frac{i\nu}{2\pi} \log b \]

\[ w(z) = \frac{dF(z)}{dz} = \frac{i\nu}{2\pi} \left( \frac{1}{z - \alpha^2/\lambda} - \frac{1}{z - 1/\lambda} - \frac{1}{z - b} \right) \quad b \to 0 \]

\[ w^2(z) = -\frac{\nu^2}{4\lambda^2} \left( \frac{1}{z - \alpha^2/\lambda} - \frac{1}{z - 1/\lambda} \right)^2 \]

\[ = -\frac{\nu^2}{4\lambda^2} \left( \frac{1}{(z - \alpha^2/\lambda)^2} - 2 \frac{1}{(z - \alpha^2/\lambda)(z - 1/\lambda)} + \frac{1}{(z - 1/\lambda)^2} \right) \]

\[ = -2 \left( \frac{\alpha^2 - 1/\lambda}{z - \alpha^2/\lambda} + \frac{1/\lambda - 1/\lambda}{z - 1/\lambda} \right) \]

\[ \Rightarrow \text{Res} \left( w^2(z) \right) = -\frac{\nu^2}{4\lambda^2} \left( -2 \right) \left( \frac{1/\lambda - 1/\lambda}{z - 1/\lambda} \right) = \frac{-\nu^2}{2\lambda^2} \frac{1/\lambda}{(1/\lambda - 1/\lambda)} \]

\[ \Rightarrow \text{Blasius Law} \Rightarrow x - i\gamma = \frac{\nu}{2} \int_{\Gamma} w^2 dz = \frac{\nu}{2} \left( 2\pi i \text{ residues of } w^2 \text{ in } \Gamma \right) \]

\[ x - i\gamma = \frac{\nu}{2} \sum_{\text{res}} \left( \frac{1/\lambda}{z - 1/\lambda} \right) \]

\[ x = \frac{\nu}{2\lambda} \frac{1/\lambda}{(1/\lambda - 1/\lambda)} = \frac{\nu}{2\lambda} \frac{1/\lambda}{(1/\lambda - 1/\lambda)} = \frac{\nu}{2\lambda} \frac{1/\lambda}{(1/\lambda - 1/\lambda)} \]

\[ \gamma = 0 \]
\[ \omega(\alpha) = \frac{P}{2\pi} \left( \frac{1}{z - \frac{a^2}{x}} - \frac{1}{z-l} \right) z = re^{i\theta} \] 

\[ = \frac{P}{2\pi} \left( \frac{1}{r e^{i\theta} - \frac{a^2}{x}} - \frac{1}{r e^{i\theta} - l} \right) \]

\[ = \frac{P}{2\pi} \left( \frac{\frac{a^2}{x} - l}{r^2 - (\frac{a^2}{x} + l) r e^{i\theta} + a^2 e^{-2i\theta}} \right) e^{-i\theta} \]

\[ = \frac{P}{2\pi} \left( \frac{\frac{a^2}{x} - l}{\frac{a^2 \cos \theta - (\frac{a^2}{x} + l)}{a}} \right) i e^{i\theta} \]

\[ = \left( u_k - i u_\theta \right) e^{-i\theta} \]

\[ u_k = 0 \]

\[ u_\theta = -\frac{P}{2\pi} \left( \frac{\frac{a^2}{x} - l}{\frac{a^2 \cos \theta - (\frac{a^2}{x} + l)}{a}} \right) \]

2) The circle of radius \( a \) is a streamline.
\[ \frac{dz}{dz} = K \left( 3-a \right)^{\frac{a}{a-1}} \left( 3-b \right)^{\frac{b}{b-1}} \left( 3-c \right)^{\frac{c}{c-1}} \]

\[ = K \left( 3-1 \right)^{-\frac{2}{2}} \left( 3 \right)^{-\frac{1}{1}} \left( 3+1 \right)^{\frac{1}{1}} \]

\[ = K \frac{\sqrt[3]{3+1}}{\sqrt[3]{3-1}} = K \frac{3+1}{3 \sqrt[3]{3^2-1}} = K \left[ \frac{1}{\sqrt[3]{3^2-1}} + \frac{1}{2 \sqrt[3]{3^2-1}} \right] \]

\[ \int dz = \int K \frac{1}{\sqrt[3]{z^2-1}} \, dz + \int K \frac{1}{\sqrt[3]{z^2-1}} \, dz \]

\[ \cosh^{-1} x = \int \frac{dx}{\sqrt{x^2-1}}, \sec^{-1} x = \int \frac{dx}{x \sqrt{x^2-1}} \]

\[ = z = K \left( \cosh^{-1} 3 + \sec^{-1} 3 \right) + D \quad \sec \pi = -1 \]

\[ z = 1 \Rightarrow z = 0 \quad \text{(A point)} \Rightarrow \cosh^{-1}(1) = 0 \quad \Rightarrow \ D = 0 \]

\[ z = -1 \Rightarrow z = (1+\lambda) \ell \quad \text{(C point)} \Rightarrow \cosh^{-1}(-1) = \pi \ell, \sec^{-1}(-1) = \pi \]

\[ \Rightarrow K (\pi \ell + \pi \ell) = (\ell+1) \ell \Rightarrow K = \frac{(\ell+1) \ell}{(\ell+1) \pi} = \frac{\ell}{\pi} \]

\[ \Rightarrow z = \frac{\ell}{\pi} \left( \cosh^{-1} 3 + \sec^{-1} 3 \right) \]
\begin{align*}
F(\zeta) &= \frac{m}{2\pi} \ln \zeta \\
W(\zeta) &= \frac{dF(\zeta)}{d\zeta} = \frac{d}{d\zeta} \frac{d}{dz} \ln \zeta \\
&= \frac{m}{2\pi \zeta} \left[ \left( \frac{\zeta}{\chi} \right)^{\frac{3+1}{2}} \right] \\
&= \frac{m}{2\pi} \left( \frac{\zeta^2 - 1}{\zeta + 1} \right) = \frac{m}{2\pi} \sqrt{\frac{\zeta - 1}{\zeta + 1}} \\
\Rightarrow \quad \frac{m}{2\pi} &= \nu \quad \text{as} \quad \zeta \to \infty \\
\Rightarrow \quad m &= 2\nu \pi \\
\Rightarrow \quad F(\zeta) &= \frac{2\nu \pi}{2\pi} \ln \zeta \\
\Rightarrow \quad F(\zeta) &= \frac{\nu}{\pi} \ln \zeta \\
\hline
\text{complex potential}
\end{align*}