Engineering Vibration

1. Introduction

The study of the motion of physical systems resulting from the applied forces is referred to as dynamics. One type of dynamics of physical systems is vibration, in which the system oscillates about certain equilibrium positions. This motion is rendered possible by the ability of materials in the system to store potential energy by their elastic properties.

Most physical systems are continuous in time and their parameters are distributed. In many cases the distributed parameters can be replaced by discrete ones by suitable lumping of the continuous system. This lumped parameter systems are described by ordinary differential equations, which are easier to solve than the partial differential equations describing continuous systems. The number of degrees of freedom specifying the number of independent coordinates necessary to define the system can be established.

Oscillatory systems can be classified into two groups according to their behavior: linear and non-linear systems. For a linear system, the principle of superposition applies, and the dependent variables in the differential equations describing the system appear to the first power only, and also without their cross products. Although only linear systems are dealt with analytically, some knowledge of non-linear systems is desirable, since all systems tend to become non-linear with increasing amplitudes of oscillation.

A physical system generally exhibits two classes of vibration: free and forced. Free vibration takes place when a system oscillates under the action of forces inherent in the system itself, and when the external forces are absent. It is described by the solution of differential equations with their right hand sides set to zero. The system when given an initial disturbance will vibrate at one or more of its natural frequencies, which are properties of the dynamical system determined by its mass and stiffness distribution. The resulting motion will be the sum of the principal modes in some proportion, and will continue in the absence of damping. Thus the mathematical study of free vibration yields information about the dynamic properties of the system, relevant for evaluating the response of the system under forced vibration.

Forced vibration takes place when a system oscillates under the action of external forces. When the excitation force is oscillatory, the system is forced to vibrate at the excitation frequency. If the frequency of excitation coincides with one of the natural frequencies, resonance is encountered, a phenomenon in which the amplitude builds up to dangerously high levels, limited only by the damping.
All physical systems are subject to one or other type of damping, since energy is dissipated through friction and other resistances. These resistances appear in various forms, after which they are named: viscous, hysteretic, Coulomb, and aerodynamic. The properties of the damping mechanisms differ from each other, and not all of them are equally amenable to mathematical formulation. Fortunately, small amounts of damping have very little influence on the natural frequencies, which are therefore normally calculated assuming no damping.
2. SINGLE DEGREE OF FREEDOM SYSTEM

2.1 Viscous Damping

A complex structure can be considered as a number of masses, interconnected by springs and damping elements to facilitate the analytical solution of the dynamic behaviour of the structure. Since the damping forces in a real structure cannot be estimated with the same accuracy as the elastic and inertia forces, a rigorous mathematical simulation of the damping effects is futile. Nevertheless, to account for the dissipative forces in the structure, assumptions of the form of damping have to be made, that give as good as possible an estimate of the damping forces in practice. Furthermore, the form has to be conducive to easy mathematical manipulation, specifically adaptable to linear equations of motion, implying that the damping forces are harmonic when the excitation is harmonic. Two such suitable forms of damping are viscous and hysteretic. The response of a single degree of freedom system to viscous damping will be described in this section and to hysteretic damping in the next section.

Fig. 1 shows a single degree of freedom system, where a mass-less dashpot of damping coefficient $c$ and a spring of stiffness $k$ are mounted between the mass $m$ and the fixed wall. The dashpot exerts a damping force $-cx$ which is proportional to the instantaneous velocity and is positive in the positive direction of $x$. The equation of motion for forced harmonic excitation may be written as

$$m\ddot{x} + c\dot{x} + kx = F_e \cos(\omega t)$$

(1)

where $x$ is the displacement
$\dot{x}$ is the velocity
$\ddot{x}$ is the acceleration
$F$ is the excitation force
\( j \) is \( \sqrt{-1} \)

and \( \omega \) is the excitation frequency.

Dividing equation (1) by \( m \) one obtain

\[
\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \omega_n^2 \left( \frac{F}{k} \right) e^{j\omega t} \tag{2}
\]

where \( \omega_n = \sqrt{\frac{k}{m}} \) undamped natural frequency,

\[
\zeta = \frac{c}{2m\omega_n} = \frac{c}{c_c} \quad \text{dimensionless damping ratio,}
\]

\( c_c \) is the critical damping.

Consider the solution of the form

\[
x = X e^{j\omega t}
\]

where \( X \) is the amplitude of the steady state vibration, it can be shown that

\[
X = \frac{\omega_n^2 F}{\omega_n^2 - \omega^2 + j2\zeta \omega \omega_n} = \frac{F / k}{1 - (\omega / \omega_n)^2 + j2\zeta \omega / \omega_n}
\tag{3}
\]

Thus

\[
x = X e^{j\omega t} = \left[ \frac{1}{1 - (\omega / \omega_n)^2 + j \zeta \omega / \omega_n} \right] \frac{Fe^{j\omega t}}{k}
\tag{4}
\]

The displacement \( x \) is proportional to the applied force, the proportionality factor being

\[
H(\omega) = \left[ \frac{1}{1 - (\omega / \omega_n)^2 + j \zeta \omega / \omega_n} \right]
\tag{5}
\]

which is known as the complex frequency response function (FRF). Equation (4) illustrates that the displacement is a complex quantity with real and imaginary parts. Thus
This shows that the displacement has one real component

\[
\text{Re}(x) = \left[ \frac{1 - (\omega / \omega_n)^2}{\left[1 - (\omega / \omega_n)^2\right]^2 + (2\zeta \omega / \omega_n)^2} \right] \frac{F e^{j\omega t}}{k}
\]  

which is in-phase with the applied force and another component

\[
\text{Im}(x) = \left[ \frac{-2\zeta \omega / \omega_n}{\left[1 - (\omega / \omega_n)^2\right]^2 + (2\zeta \omega / \omega_n)^2} \right] \frac{F e^{j\omega t}}{k}
\]  

which has a phase lag of 90° behind the applied force.

The amplitude of the total displacement is given by

\[
\sqrt{\text{Re}^2(X) + \text{Im}^2(X)} = \frac{1}{\sqrt{\left[1 - (\omega / \omega_n)^2\right]^2 + (2\zeta \omega / \omega_n)^2}} \frac{F e^{j\omega t}}{k}
\]

The total displacement lags behind the force vector by an angle θ given by

\[
\theta = \tan^{-1} \frac{\text{Im}(x)}{\text{Re}(x)}
\]

The steady state solution of Eq.(2) can therefore also be written in the form

\[
x = \left[ \frac{1}{\sqrt{\left[1 - (\omega / \omega_n)^2\right]^2 + (2\zeta \omega / \omega_n)^2}} \right] \frac{F e^{j(\omega t - \theta)}}{k}
\]

where θ is given by Eq. (10).

The quantity in the square brackets of Eq. (11) is the absolute value of the complex frequency response, \(|H(\omega)|\), and it is called the magnification factor, a
dimensionless ratio between the amplitude of displacement $X$ and the static displacement $F/k$.

Fig. 2(a) Magnification factor $|H(\omega)|$ as a function of the dimensionless frequency ratio $\omega/\omega_n$ for various damping ratio $\zeta$, and (b) phase lag of displacement behind force as a function of $\zeta$. 
Fig. 2(a) shows the absolute value of the complex frequency response function 

$$|H(\omega)|$$

as a function of the dimensionless frequency ratio $\omega / \omega_n$ for various damping ratio $\zeta$. It can be seen that increasing damping ratio $\zeta$ tends to diminish the amplitudes and to shift the peaks to the left of the vertical through $\omega / \omega_n = 1$. The peaks occur at frequencies given by

$$\omega = \omega_n \sqrt{1 - 2\zeta^2}$$  \hspace{1cm} (12)

where the peak value of $|H(\omega)|$ is given by

$$|H(\omega)| = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}$$  \hspace{1cm} (13)

For light damping ($\zeta < 5\%$) the curves are nearly symmetric about the vertical through $\omega / \omega_n = 1$. The peak value of $|H(\omega)|$ occurs in the immediate vicinity of $\omega / \omega_n = 1$ and is given by

$$|H(\omega)| = \frac{1}{2\zeta} = Q$$  \hspace{1cm} (14)

where $Q$ is known as the quality factor.

Fig. 2(b) shows the phase angle $\theta$ against $\omega / \omega_n$ for various $\zeta$ plotted from Eq. (10). It should be noted that all curves pass through the point $\theta = 90^\circ (\pi / 2)$, $\omega / \omega_n = 1$; i.e., no matter what the damping is, the phase angle between force and displacement at the undamped natural frequency $\omega / \omega_n$ is $90^\circ$. Moreover, the phase angle tends to zero for $\omega / \omega_n \rightarrow 0$ and to $180^\circ$ for $\omega / \omega_n \rightarrow \infty$.

Consider the system with $\zeta = 10\%$ for example, Fig. 3(a) defines the points $P_1$ and $P_2$ where the
Fig. 3 (a) Magnification factor $|H(\omega)|$ as a function of the dimensionless frequency ratio $\omega/\omega_n$ for $\zeta = 10\%$, and (b) complex $s$-plane of the poles.

Amplitude of $|H(\omega)|$ reduces to $Q/\sqrt{2}$ of its peak value are called the half power points. If the ordinate is plotted on a logarithmic scale, $P_1$ and $P_2$ are points where the amplitude of $|H(\omega)|$ reduces by 3 dB and are thus called the -3 dB points. The difference in the frequencies of points $P_1$ and $P_2$ is called the 3 dB bandwidth of the system, and for light damping it can be shown that

$$\Delta \omega = \omega_2 - \omega_1 = 2\zeta \omega_n$$

(15)

where $\Delta \omega$ is the 3 dB bandwidth

$\omega_1$ is the frequency at point $P_1$

$\omega_2$ is the frequency at point $P_2$

From Eqs. (14) and (15)

$$\frac{\omega_2 - \omega_1}{\omega_n} = 2 \frac{c}{c_c} = \frac{1}{Q} = \eta$$

(16)

where $\eta$ is called the Loss Factor.

To examine the variation of the in-phase Re($x$) and quadrature Im($x$) components of displacement, Eqs. (7) and (8) are plotted as a function of in Figs. 4(a) and 4(b) respectively. The curves of the real component of displacement in Fig. 4(a) have a
zero value at $\omega / \omega_n = 1$ independent of damping ratio $\zeta$, and exhibits a peak and a notch at frequencies

$$
\omega_1 = \omega_n \sqrt{1 - 2\zeta} \\
\omega_2 = \omega_n \sqrt{1 + 2\zeta}
$$

As the damping decreases, the peak and the notch increase in value and become closer together. In the limit when $\zeta = 0$, the curve has an asymptote at $\omega / \omega_n = 1$. The frequencies $\omega_1$ and $\omega_2$ are often used to determine the damping of the system from the equation

$$
\eta = 2\zeta = \frac{(\omega_2 / \omega_1)^2 - 1}{(\omega_2 / \omega_1)^2 + 1}
$$

The curves of the imaginary component of displacement have a notch in close vicinity of $\omega / \omega_n = 1$ and they are sharper than those of $|H(\omega)|$ in Fig. 2(a) for corresponding values of $\zeta$. 

![Graph](image)
Fig. 4. (a) Real component of displacement as a function of the dimensionless frequency ratio for various values of $\zeta$, and (b) imaginary component of displacement against for various values of $\zeta$. 
2.2 Hysteretic (structural) Damping

Another type of damping which permits setting up of linear damping equation, and which may often give a closer approximation to the damping process in practice, is the hysteretic damping, sometimes called structural damping. A large variety of materials, when subjected to cyclic stress (for strains below the elastic limit), exhibit a stress-strain relationship which is characterized by a hysteresis loop. The energy dissipated per cycle due to internal friction in the material is proportional to the area within the hysteresis loop, and hence the name hysteretic damping. It has been found that the internal friction is independent of the rate of strain (independent of frequency) and over a significant frequency range is proportional to the displacement. Thus the damping force is proportional to the elastic force but, since energy is dissipated, it must be in phase with the velocity (in quadrature with displacement).

Thus for simple harmonic motion the damping force is given by

\[ j\gamma k x = \gamma \frac{\ddot{x}}{\omega} \]  

(19)

where \( \gamma \) is called the structural damping factor. The equation of motion for a single degree of freedom system with structural damping can thus be written

\[ m\ddot{x} + \frac{\gamma k}{\omega} \dot{x} + kx = F_e^{j\omega} \]  

(20)

\[ m\ddot{x} + k(1 + j\gamma) = F_e^{j\omega} \]  

(21)

where \( \gamma \) is called the complex stiffness.

The steady state solution of Eq. (21) is given by

\[ x = X e^{j\omega t} = \left[ \frac{1}{1 - (\omega / \omega_n)^2 + j\gamma} \right] \frac{F e^{j\omega t}}{k} \]  

(22)

corresponding to Eq. (4) for viscous damping.

The real and the imaginary components of the displacement can be obtained:

\[ x = \left[ \frac{1 - (\omega / \omega_n)^2}{\left\{1 - (\omega / \omega_n)^2\right\}^2 + \gamma^2} - \frac{j\gamma}{\left\{1 - (\omega / \omega_n)^2\right\}^2 + \gamma^2} \right] \frac{F e^{j\omega t}}{k} \]  

(23)
Thus
\[ \text{Re}(x) = \frac{1 - (\omega/\omega_n)^2}{\left(1 - (\omega/\omega_n)^2\right)^2 + \gamma^2} \frac{F e^{i\omega t}}{k} \] (24)

and
\[ \text{Im}(x) = \frac{-\gamma}{\left(1 - (\omega/\omega_n)^2\right)^2 + \gamma^2} \frac{F e^{i\omega t}}{k} \] (25)

The total displacement is given by
\[ x = \left[ \frac{1}{\sqrt{\left(1 - (\omega/\omega_n)^2\right)^2 + \gamma^2}} \right] \frac{F e^{i\omega t}}{k} \] (26)

which lags behind the force vector by an angle \( \theta \) given by
\[ \theta = \tan^{-1} \left( \frac{\gamma}{1 - (\omega/\omega_n)^2} \right) \] (27)
Fig. 5 (a) Magnification factor as a function of $\omega/\omega_n$ for various values of the structural damping factor $\gamma$, and (b) phase lag of displacement behind force as a function of $\omega/\omega_n$ for various values of $\gamma$.

The term in the square brackets of expression (26) (magnification factor) and $\theta$ are plotted against $\omega/\omega_n$ for various values of $\gamma$ in Figs. 5(a) and (b) respectively. The curves of Figs. 5(a) and (b) can be seen to be similar to those of Figs. 2(a) and 2(b) respectively for viscous damping; however, there are some minor differences. For hysteretic damping it can be seen from Fig. 5(a) that the maximum response occurs exactly at $\omega/\omega_n = 1$ independent of damping $\gamma$. At very low values of $\omega/\omega_n$ the response for hysteretic damping depends on $\gamma$ and the phase angle $\theta$ tends to $\tan^{-1}\gamma$ whereas it is zero for viscous damping.
2.3 Complex Frequency Solution

It has been shown that the steady state solution to Eq.(2) for forced vibration of such a system was given by \( x = Xe^{j\omega t} \). In the case of free damped vibration, for which the equation of motion is given by

\[
mx'' + cx' + kx = 0
\]

the above solution can be extended to a more general type of the form

\[
x = Xe^{\sigma t}
\]

where \( s = \sigma + j\omega \) is known as the complex frequency.

Before solving Eq.(3.1) it is important to examine the variable \( s \). Since \( s \) is complex it is best illustrated the complex frequency plane, where the real axis represents \( \sigma \) the decay rate (amount of damping), and the imaginary axis represents \( j\omega \), the frequency. Every point in this plane defines a particular form of oscillation. For positive \( \sigma \) the magnitude increases exponentially with time, and for negative \( \sigma \) it decreases exponentially with time. Values of \( s \) which are complex, thus lying anywhere in the plane, giving solutions of the form \( x = Xe^{(\sigma + j\omega)t} \) describe oscillatory motion increasing or decreasing exponentially with time depending on the sign of \( \sigma \).

The anticipated solution (3.2) is inserted in (3.1), leading to the algebraic equation

\[
\frac{s^2}{k}m + \frac{s}{k}c + 1 = 0
\]

which is known as the characteristic equation of the system. In the absence of damping the natural frequency of the system is given by \( \omega_n = \sqrt{k/m} \) which may be substituted in Eq. (3.3) to yield
\[
\left( \frac{s}{\omega_n} \right)^2 + \frac{c}{\sqrt{km}} \left( \frac{s}{\omega_n} \right) + 1 = 0
\]  
(3.4)

or

\[
\left( \frac{s}{\omega_n} \right)^2 + 2\zeta \left( \frac{s}{\omega_n} \right) + 1 = 0
\]  
(3.5)

where \( \zeta = \frac{c}{c_c} = \frac{1}{2 \sqrt{km}} \) is the damping ratio.

Since Eq. (3.5) is a quadratic, the two roots of the equation are given by

\[
\left( \frac{s_{1,2}}{\omega_n} \right)^2 = -\zeta \pm \sqrt{\zeta^2 - 1}
\]  
(3.6)

and the values of \( s \) obtained, thus define two forms of oscillation in the complex plane. Obviously the nature of the roots \( s_1 \) and \( s_2 \) depends on the value of \( \zeta \). The effect of the variation of \( \zeta \) on the roots can be illustrated in the complex plane (s-plane) in the form of a diagram representing the locus of roots plotted as a function of the parameter \( \zeta \). From Eq. (3.6) it can be seen, that for \( \zeta = 0 \) the roots are given by \( s_{1,2} = \pm j\omega_n \), which lie on the imaginary axis, corresponding to undamped oscillations at the natural frequency of the system.

For \( 0 < \zeta < 1 \) or \( c < 2 \sqrt{km} \) which is known as the underdamped case, the roots are given by

\[
\left( \frac{s_{1,2}}{\omega_n} \right)^2 = -\zeta \pm j\sqrt{1-\zeta^2}
\]  
(3.7)

Thus \( s_1 \) and \( s_2 \) are always complex conjugate pairs, located symmetrically with respect to the real axis. As the values of \( \left( \frac{s}{\omega_n} \right) \) are complex, the magnitude

\[
\left| \frac{s}{\omega_n} \right| = \sqrt{\zeta^2 + 1 - \zeta^2} = 1
\]  
(3.8)

indicating that the locus of the roots is a circle of radius \( \omega_n \), centered at the origin. Furthermore, when the complex conjugate pairs of the roots are associated with each other, they can be interpreted as a real oscillation, for example
\[ e^{(-\sigma + j\omega)t} + e^{(-\sigma - j\omega)t} = 2e^{-\sigma t}\cos\omega t \] (3.9)

As \( \zeta \) approaches unity, \( (c \to 2\sqrt{km}) \) the two roots approach the point \( -\omega_n \) on the real axis, which is known as the critically damped case.

For \( \zeta > 1 \) or \( c > 2\sqrt{km} \) known as the overdamped case, the roots are given by

\[ s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \] (3.10)

which lie on the negative real axis. As \( \zeta \to \infty, s_1 \to 0 \) and \( s_2 \to -\infty \).

To obtain the time response of free vibration for given initial conditions, Eq.(3.6) is substituted in Eq.(3.2) to give

\[ x(t) = X_1e^{s_1t} + X_2e^{s_2t} \] (3.11)

For the overdamped case \( \zeta > 1 \), the solution is given by

\[ x(t) = X_1e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + X_2e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \]

i.e.

\[ x(t) = \left\{ X_1e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + X_2e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \right\} e^{-\zeta\omega_n t} \] (3.12)

which describes aperiodic motion decaying exponentially with time. \( X_1 \) and \( X_2 \) are determined from the initial displacement and velocity which in turn govern the shape of the decaying curve.

For the critically damped case \( \zeta = 1 \), Eq. (3.5) has a double root \( s_1 = s_2 = -\omega_n \) and the solution is given by

\[ x(t) = (X_1 + tX_2)e^{-\omega_n t} \] (3.13)

The response again represents aperiodic motion decaying exponentially with time. However, a critically damped system approaches the equilibrium position the fastest for given initial conditions.

For the underdamped case \( 0 < \zeta < 1 \) the solution is given by

\[ x(t) = \left\{ X_1e^{j\sqrt{\zeta^2 - 1}\omega_n t} + X_2e^{-j\sqrt{\zeta^2 - 1}\omega_n t} \right\} e^{-\zeta\omega_n t} \]
i.e. \[ x(t) = \left\{ X_1 e^{i\omega_d t} + X_2 e^{-i\omega_d t} \right\} e^{-\xi \omega_d t} \] (3.14)

where \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \) is called the frequency of damped free vibration. Eq.(3.14) represents exponentially decaying oscillatory motion with constant frequency \( \omega_d \).
3. MULTI–DEGREE OF FREEDOM SYSTEMS

3.1 FREE VIBRATION

The number of degrees of freedom chosen dictates the number of differential equations necessary to characterize the system. As these equations are normally coupled to each other, they must be decoupled before their solution is attempted. To do this, the orthogonal properties of the Principal Modes are exploited, enabling the original differential equations to be rewritten in terms of the principal coordinates.

However, in calculating the response under forced vibration, not only is it necessary to make assumptions about the type of damping, but also the distribution of damping--proportional or non-proportional. This is because the latter type of distribution, generally encountered in complex structures, is considerably more difficult to resolve mathematically, giving what is called complex modes, in contrast to real modes obtained with proportional damping.

3.1.1 Natural Frequency and Vibration Mode

The equations of motion of the system shown in Fig.9 are

\[
\begin{align*}
    m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 - c_2 \dot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2 &= F_1 \\
    m_2 \ddot{x}_2 - c_2 \dot{x}_1 + (c_2 + c_3) \dot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 &= F_2
\end{align*}
\]

which can be written in matrix form as

\[
\begin{bmatrix}
    m_1 & 0 \\
    0 & m_2
\end{bmatrix}
\begin{bmatrix}
    \ddot{x}_1 \\
    \ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
    c_1 + c_2 & -c_2 \\
    -c_2 & c_2 + c_3
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
    k_1 + k_2 & -k_2 \\
    -k_2 & k_2 + k_3
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}
= \begin{bmatrix}
    F_1 \\
    F_2
\end{bmatrix}
\]

To determine the natural frequencies and natural mode shapes of the system, the undamped free vibration of the system is first considered. Thus the equations reduce to
\[ [m][\ddot{x}] + [k][x] = 0 \]  \hspace{1cm} (3.3)

where

\[
[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \text{and} \quad [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}
\]

Assuming harmonic motion \( \{x\} = \{v\}e^{j\omega t} \) with \( \lambda = \omega^2 \) Eq. (3.3) becomes

\[-\lambda [m]\{x\} + [k]\{x\} = 0\]

or

\[\left[ (-\lambda [m] + [k]) \right]\{v\} = 0 \quad (3.4)\]

Premultiplying Eq. (4.4) by \( [m]^{-1} \) and rearranging we obtain

\[\left[ [m]^{-1}[k] - \lambda [I] \right]\{v\} = 0 \quad (3.5)\]

where \( [m]^{-1} \) \( [k] \) is called a dynamic matrix and \( [m]^{-1} [m] = [I] \) is a unit matrix.

Eq. (3.5) is a set of simultaneous algebraic equations in \( \{v\} \). From the theory of equations it is known, that for a non-trivial solution \( \{v\} = 0 \), the determinant of the coefficients of Eq. (3.5) must be zero. Thus

\[\left| [m]^{-1}[k] - \lambda [I] \right| = 0 \quad (3.6)\]

which is known as the characteristic equation of the system. Eq. (3.6) when expanded can be rewritten as

\[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \quad (3.7)\]

which is a polynomial in \( \lambda \) for an \( n \)-degree-of-freedom system. The roots \( \lambda_i \) of the characteristic equation are called eigenvalues and the undamped natural frequencies of the system are determined from the relationship

\[\lambda_i = \omega_i^2 \quad (3.8)\]
By substituting \( \lambda_i \) into matrix equation (3.5) we obtain the corresponding natural (or Principal) mode shape \( \{v_i\} \) which is also called an eigenvector. The mode shape represent a deformation pattern of the structure for the corresponding natural frequency. As Eq. (3.5) are homogeneous, there is not a unique solution for the \( \{v\} \) s. Thus the natural mode shape is defined by the ratio of the amplitudes of motion at the various points on the structure when excited at its natural frequency. The actual amplitude on the other hand depends on the initial conditions and the position and magnitude of the exciting forces.

Consider a numerical example for the system shown in Fig.9 where

\[
m_1 = 5 \text{kg}; m_2 = 10 \text{kg}; k_1 = k_2 = 2N/m; k_3 = 4N/m
\]

Substituting in Eq. (3.2) we get

\[
\begin{bmatrix}
5 & 0 \\
0 & 10
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix}
+
\begin{bmatrix}
4 & -2 \\
-2 & 6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Thus Eq. (3.5) becomes

\[
\begin{bmatrix}
5 & 0 \\
0 & 10
\end{bmatrix}
^{-1}
\begin{bmatrix}
4 & -2 \\
-2 & 6
\end{bmatrix}
\lambda
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
4/5 - \lambda & -2/5 \\
-1/5 & 3/5 - \lambda
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

For a non-trivial solution the determinant of the above equation must equal zero, thus the characteristic equation

\[
\lambda^2 - \frac{7}{5} \lambda + \frac{2}{5} = 0
\]

The roots of the above equation are

\[
\lambda_{1,2} = \frac{2}{5} \quad \text{and} \quad 1
\]

Thus \( \lambda_1 = \frac{2}{5} \) and \( \lambda_2 = 1 \) and the two natural frequencies are given by
\[ \omega_1 = \sqrt{\lambda_1} = \sqrt{2/5} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = 1 \]

Substitution of \( \lambda_1 \) and \( \lambda_2 \) in Eq. (3.10) will give the two natural mode shapes. Thus the mode shape for the natural frequency \( \omega_1 \) is \( \{v_1\} = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} \) where \( v_{11} \) is arbitrary.

Similarly the mode shape for the natural frequency \( \omega_2 \) is \( \{v_2\} = \begin{pmatrix} v_{21} \\ -v_{21}/2 \end{pmatrix} \)

For an arbitrary deflection of \( v_{11} = v_{21} = 1 \) the two mode shapes would be

\[ \{v_1\} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for} \quad \omega_1 = \sqrt{2/5} \]

and

\[ \{v_2\} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \quad \text{for} \quad \omega_2 = 1 \]

For \( \omega_1 = \sqrt{2/5} \) rad/s the masses move in - phase

For \( \omega_2 = 1 \) rad/s the masses move out - of - phase

Fig. 10. Frequency and mode shapes for the two-degree-of-freedom system.

The system can thus vibrate freely with simple harmonic motion when started in the correct way at one of two possible frequencies as shown in Fig.10. Note that the masses move either in phase or \( 180^\circ \) out of phase with each other. Since the masses reach their maximum displacements simultaneously, the nodal points are clearly defined.
3.1.2 Orthogonal Properties of Eigenvectors

It was shown in the previous section that solution of Eq. (3.4) yields \( n \) eigenvalues and \( n \) corresponding eigenvectors. Thus a particular eigenvalue \( \omega_i \) and the eigenvector \( \{v_i\} \) will satisfy Eq. (3.4); i.e.,

\[
[k]\{v_i\} = \lambda_i [m]\{v_i\} \tag{3.11}
\]

Premultiply Eq. (3.11) by the transpose of another mode shape \( \{v_j\} \), i.e.,

\[
\{v_j\}^T[k]\{v_i\} = \lambda_i \{v_j\}^T[m]\{v_i\}, \text{ where the superscript } T \text{ denotes a transpose matrix.}
\]

We now write the equation for the \( j^{th} \) mode and premultiply by the transpose of the \( i^{th} \) mode, i.e.,

\[
\{v_i\}^T[k]\{v_j\} = \lambda_i \{v_i\}^T[m]\{v_j\} \tag{3.12}
\]

As \([k]\) and \([m]\) are symmetric matrices

\[
\{v_j\}^T[k]\{v_i\} = \{v_j\}^T[k]\{v_i\} \tag{3.13}
\]

and

\[
\{v_j\}^T[m]\{v_i\} = \{v_j\}^T[m]\{v_i\} \tag{3.14}
\]

Therefore subtracting Eq. (3.13) from Eq. (3.12) we obtain

\[
0 = (\lambda_i - \lambda_j) \{v_i\}^T[m]\{v_j\} \tag{3.15}
\]

If \( \lambda_i \neq \lambda_j \) (implying two different natural frequencies)

\[
\{v_i\}^T[m]\{v_j\} = 0 \tag{3.16}
\]

and from Eq. (3.13) it can be seen that

\[
\{v_i\}^T[k]\{v_j\} = 0 \tag{3.17}
\]
Equations (3.16) and (3.17) define the orthogonality properties of the mode shapes with respect to the system mass and stiffness matrices respectively.

### 3.1.3 Generalized Mass and Generalized Stiffness

It can be seen that if \( i = j \) in Eq. (3.15) then the two modes are not necessarily orthogonal and Eq. (3.16) is equal to some scalar constant other than zero, e.g. \( m_i \)

\[
\{v_i\}^T \left[ m \right] \{v_i\} = m_i \quad i = 1,2,3,...n \tag{3.18}
\]

and from Eq. (3.13) it follows that

\[
\{v_i\}^T \left[ k \right] \{v_i\} = \lambda_i m_i = \omega_i^2 m_i = k_i \quad i = 1,2,3,...n \tag{3.19}
\]

\( m_i \) and \( k_i \) are called the generalized mass and generalized stiffness respectively.

The numerical values of the mode shapes calculated above will be used to determine the generalized mass and generalized stiffness. The mode shapes were found to be

\[
\{v_1\} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for } \omega_1 = \sqrt{2/5} \quad \text{and} \quad \{v_{1/2}\} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \text{ for } \omega_2 = 1
\]

Substituting \( \{v_1\} \) in Eq. (3.18) we obtain the generalized mass \( m_1 \) for the first mode: \( m_1 = 15 \). Similarly substituting \( \{v_2\} \) in Eq. (3.18) we obtain \( m_2 = 15/2 \).

Thus the generalized masses \( m_1 \) and \( m_2 \) for the first and second modes are 15 and 15/2 respectively. The generalized stiffnesses \( k_1 \) and \( k_2 \) for the first and second modes are \( k_1 = \omega_1^2 m_1 = 6 \) and \( k_2 = \omega_2^2 m_2 = 2/15 \).

### 3.1.4 Normal Mode

If one of the elements of the eigenvector \( \{v_i\} \) is assigned a certain value, the rest of the \( (n-1) \) elements are also fixed because the ratio between any two elements is constant. Thus the eigenvector becomes unique in an absolute sense. This process of adjusting the elements of the natural modes to make their amplitude unique is called normalization, and the resulting scaled natural modes are called orthonormal modes, or normal modes. There are several ways to normalize the mode shapes, and the
common practice in engineering is that the mode shapes can be normalized such that
the generalized mass or modal mass $m_i$ in Eq. (3.18) is set to unity. This method
has the advantage that Eq. (3.19) yields directly the eigenvalues $\lambda_i$ and thus the natural
frequencies.

The normalization will be illustrated by the numerical example of the system
shown in Fig.9. The mode shapes were shown to be

$$\{v_1\} = \begin{bmatrix} v_{11} \\ v_{11} \end{bmatrix} \quad \text{for} \quad \omega_1 = \sqrt{2/5} \quad \text{and} \quad \{v_2\} = \begin{bmatrix} v_{21} \\ -v_{21}/2 \end{bmatrix} \quad \text{for} \quad \omega_2 = 1$$

Therefore substitution of $\{v_1\}$ in Eq. (3.18) yields

$$\begin{bmatrix} v_{11} \\ v_{11} \end{bmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{11} \end{bmatrix} = m_1 = 1$$

And the normalized mode shape for $\omega_1 = \sqrt{2/5}$ is

$$\{v_1\} = \begin{bmatrix} v_{11} \\ v_{11} \end{bmatrix} = \begin{bmatrix} \sqrt{1/15} \\ \sqrt{1/15} \end{bmatrix} \quad (3.20)$$

Similarly for $\{v_2\}$ we get

$$\begin{bmatrix} v_{21} \\ -v_{21}/2 \end{bmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} v_{21} \\ -v_{21}/2 \end{bmatrix} = m_2 = 1 ,$$

and the normalized mode shape for $\omega_2 = 1$ is

$$\{v_2\} = \begin{bmatrix} v_{21} \\ -v_{21}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2/15} \\ -\sqrt{2/5/15} \end{bmatrix} \quad (3.21)$$

It can now be seen that these normalized mode shapes could also have been
obtained by dividing the natural modes by the square root of their respective
generalized masses calculated in the previous section, i.e., for normalization of the
first natural mode
\[ \{v_1\} = \frac{1}{\sqrt{m_1}} \{1\} = \frac{1}{\sqrt{15}} \{1\} = \left[ \frac{\sqrt{1/15}}{\sqrt{1/15}} \right] \]

and for the second mode
\[ \{v_2\} = \frac{1}{\sqrt{m_2}} \left\{ \begin{array}{c} 1 \\ -1/2 \end{array} \right\} = \frac{1}{\sqrt{15/2}} \left\{ \begin{array}{c} 1 \\ -1/2 \end{array} \right\} = \left[ \frac{\sqrt{2/15}}{-\sqrt{2/15}/2} \right] \]

which are the same as calculated in Eq. (3.20) and (3.21) respectively.

### 3.2 Forced Vibration

The equations of motion of the two-degree-of-freedom system shown in Fig. 9 without damping can be written as

\[
\begin{align*}
    m_1 \ddot{x}_1 &+ (k_1 + k_2) x_1 - k_2 x_2 = F \\
    m_2 \ddot{x}_2 &- k_2 x_1 + (k_2 + k_3) x_2 = l
\end{align*}
\]

or in matrix form as

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2
\end{bmatrix} +
\begin{bmatrix}
  (k_1 + k_2) & -k_2 \\
  -k_2 & (k_2 + k_3)
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} =
\begin{bmatrix}
  F_1 \\
  F_2
\end{bmatrix}
\]

In solving the above equations for the response \( \{x\} \) for a particular set of exciting forces, the major obstacle encountered is the coupling between the equations; i.e., both coordinates \( x_1 \) and \( x_2 \) occur in each of the Eq. (3.22). In Eq. (3.23) the coupling is seen by the fact that while the stiffness matrix is symmetric, it is not diagonal (i.e., the off-diagonal terms are non-zero). This type of coupling is called elastic coupling or static coupling (non-diagonal stiffness matrix) and occurs for a lumped mass system, if the coordinates chosen are at each mass point. If the equations of motion had been written in terms of the extensions of each spring, the stiffness matrix would have been diagonal but not the mass matrix. This kind of coupling is termed inertial coupling or dynamic coupling (non-diagonal mass matrix). It is thus seen that the way in which the equations are coupled depends on the choice of coordinates. If the system of equations could be uncoupled, so that we obtained diagonal mass and stiffness matrices, then each equation would be similar to that of a single degree of freedom system, and could be solved independent of each other. Indeed, the process of deriving the system response by transforming the equations of motion into an independent set of equations is known as modal analysis.
Thus the coordinate transformation we are seeking, is one that decouples the system inertially and elastically simultaneously, and therefore yields us diagonal mass and stiffness matrices. It is here that the orthogonal properties of the mode shapes discussed above come into use. It was shown by Eq. (3.18), that if the mass or the stiffness matrix is post and pre-multiplied by a mode shape and its transpose respectively, the result is some scalar constant. Thus with the use of a matrix $[v]$ whose columns are the mode shape vectors, we already have at our disposal the necessary coordinate transformation. The $x$ coordinates are transformed to $\eta$ by the equation

$$\{x\} = [v]\{p\}$$  \hspace{1cm} (3.24)

where

$$[v] = \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{1n} \\ v_{12} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1n} & v_{2n} & \cdots & v_{nn} \end{bmatrix}$$  \hspace{1cm} (3.25)

$[v]$ is referred to as the modal matrix and $\{p\}$ is called principal coordinates, normal coordinates or modal coordinates. Eq. (3.23) can be written as

$$[m]\{x\} + [k]\{x\} = \{F\}$$ \hspace{1cm} (3.26)

and substituting Eq. (3.24) into (3.26) yields

$$[m][v]\{\dot{p}\} + [k][v]\{p\} = \{F\}$$ \hspace{1cm} (3.27)

Pre-multiplying Eq. (3.26) by the transpose of the modal matrix, i.e, $[v]^T$, we obtain

$$[v]^T[m][v]\{\dot{p}\} + [v]^T[k][v]\{p\} = [v]^T\{F\}$$ \hspace{1cm} (3.28)

In Eq. (3.18) the mass matrix was post and pre-multiplied by one mode shape and its transpose, giving a scalar quantity, while in Eq. (3.28) the mass matrix is post and premultiplied by all the mode shapes and their transpose. Thus the product is a matrix
\[ [M] \text{ whose diagonal elements are some constants while all the off-diagonal terms are zero, i.e.} \]

\[ [v]^T [m][v] = [v] m_i \]  \hspace{1cm} (3.29)

Similarly

\[ [\phi]^T [m][\phi] = [\phi] k_i \]  \hspace{1cm} (3.30)

where \([v] m_i\) and \([\phi] k_i\) are diagonal matrices.

Hence Eq. (3.28) can be written as

\[ [v] m_i \{ \ddot{\theta} \} + [\phi] k_i \{ p \} = [v] F \]  \hspace{1cm} (3.31)

Eq. (3.31) represents \(n\)-equations of the form

\[ m_i \ddot{\theta}_i + k_i p_i = \{ v_i \}^T \{ F \} = f_i \]  \hspace{1cm} (3.32)

where \(\{ v_i \}\) is the \(i^{th}\) column of the modal matrix, i.e., the \(i^{th}\) mode shape. \(m_i\) and \(k_i\) can be recognised as the \(i^{th}\) modal mass (generalized mass) and \(i^{th}\) modal stiffness (generalized stiffness) respectively. Eq. (3.32) is the equation of motion for single degree of freedom systems shown in Fig.11.

Fig. 11. Single degree of freedom system defined by Eq. (3.32).

Since \(k_i = \omega_i^2 m_i\), Eq. (3.32) can be written as
\[ p_i + \omega_i^2 p_i = \frac{f_i}{m_i} \left\{ v \right\}_i^T \left\{ F \right\} \]

(3.33)

Once the solution (time responses) of Eq. (3.31) for all \( p_i \) is obtained, the solution in terms of the original coordinates \( \{x\} \) can be obtained by transforming back, i.e. substituting for \( \eta_i \) in Eq. (3.24) \( \{x\} = [\{v\}] [\{p\}] \). It should be noted that when the modal matrix \( [v] \) of Eq. (3.25) is made up of columns of the normalized mode shapes (such that \( m_i = 1 \)). Eq. (3.33) would be simplified to

\[ p_i + \omega_i^2 p_i = \left\{ v \right\}_i^T \left\{ F \right\} \]

(3.34)

Thus the modal mass would be unity and the modal stiffness would be the square of the natural frequency of the \( i^{th} \) mode.

Let us consider our numerical example of the system of Fig.9 with forcing functions \( F_1 \) and \( F_2 \). Thus Eq. (4.9) becomes

\[
\begin{bmatrix}
5 & 0 \\
0 & 10
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
4 & -2 \\
-2 & 6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\]

(3.35)

The natural frequencies and natural mode shapes were

\[ \omega_1 = \sqrt{2/5} \quad \{v_1\} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ \omega_2 = 1 \quad \{v_2\} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \]

Thus the modal matrix \( [v] \) using natural mode shapes

\[
[v] = \begin{bmatrix} 1 & 1 \\ 1 & -1/2 \end{bmatrix}
\]

The \( x \) coordinates are transformed by the equation
\[ \{x\} = \{v\} \{p\} \] (3.36)

i.e.
\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1/2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \] (3.37)

Substituting Eq. (4.41) into Eq. (4.40) and pre-multiplying by \( \{v\}^T \) gives
\[
\begin{pmatrix} v \\ 0 \\ 10 \end{pmatrix} \begin{pmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} v \\ p_1 \\ p_2 \end{pmatrix} = \{v\}^T \{F_1, F_2\} \]

The products \( \{v\}^T \{m\} \{v\} \) and \( \{v\}^T \{k\} \{v\} \) are calculated to be
\[
\begin{pmatrix} v \\ 0 \\ 10 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} v \\ \ddot{p}_1 \\ \ddot{p}_2 \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \end{pmatrix}
\]
\[
\begin{pmatrix} v \\ 0 \\ 10 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} v \\ \ddot{p}_1 \\ \ddot{p}_2 \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}
\]

Substituting these products into the equation above we obtain
\[
\begin{pmatrix} 15 & 0 \\ 0 & 15/2 \end{pmatrix} \begin{pmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 15/2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}
\]

Thus the equations of motion in are
\[
\begin{cases}
15 \ddot{p}_1 + 6 \ddot{p}_2 = F_1 + F_2 \\
15/2 \ddot{p}_2 + 15/2 \ddot{p}_2 = F_1 - F_2 / 2
\end{cases}
\] (3.38)

The original set of equations (3.38) are shown to be uncoupled; in other words the two degree of freedom system is broken down to two single degree of freedom systems shown in Fig.12.
Fig. 12. The undamped two-degree-of-freedom system shown in Fig. 9, is transformed into two single degree of freedom systems.

As mentioned above, the modal matrix can also be made up of columns of the normalized mode shapes (such that $m_i = 1$). Using the normalized mode shapes from Eqs. (3.20) and (3.21), the normalized modal matrix $[v]$ is given by

$$[v] = \begin{bmatrix} \sqrt{1/15} & \sqrt{2/15} \\ \sqrt{1/15} & -\sqrt{2/15}/2 \end{bmatrix}$$

Thus the products $[v]^T [m][v]$ and $[v]^T [k][v]$ are given by

$$[v]^T [m][v] = \begin{bmatrix} \sqrt{1/15} & \sqrt{1/15} \\ \sqrt{2/15} & -\sqrt{2/15}/2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1/15} & \sqrt{2/15} \\ \sqrt{1/15} & -\sqrt{2/15}/2 \end{bmatrix} = \begin{bmatrix} 2 \sqrt{1/5} \\ \sqrt{1/5} \end{bmatrix}$$

$$[v]^T [k][v] = \begin{bmatrix} \sqrt{1/15} & \sqrt{1/15} \\ \sqrt{2/15} & -\sqrt{2/15}/2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \sqrt{1/15} & \sqrt{2/15} \\ \sqrt{1/15} & -\sqrt{2/15}/2 \end{bmatrix} = \begin{bmatrix} 2 \sqrt{1/5} \\ \sqrt{1/5} \end{bmatrix}$$

It can thus be seen that, by using the normalized modal matrix for coordinate transformation, the mass matrix becomes a unit matrix and the stiffness matrix is diagonalized with the diagonal terms equal to the eigenvalues (the square of the natural frequencies).
In general
\[ [v]^T[m][v] = [1] \]
and
\[ [v]^T[k][v] = [\lambda] \]
where
\[
\begin{bmatrix}
\lambda_1 & 0 & 0 & \ldots & 0 \\
0 & \lambda_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n
\end{bmatrix}
\]

Once the time responses for \( p_1 \) and \( p_2 \) have been determined from Eq. (3.38), they can be substituted in Eq. (3.37) to give the time response in terms of the original coordinates \( x \), thus
\[
\begin{cases}
  x_1(t) = p_1(t) + p_2(t) \\
  x_2(t) = p_1(t) - \frac{1}{2} p_2(t)
\end{cases}
\]
(3.39)

Eq. (3.39) in fact illustrates a very important principle in vibration, namely that any possible free motion can be written as the sum of the motion in each principal mode in some proportion and relative phase. In general for an \( n \)-degree-of-freedom system
\[
\begin{cases}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{cases}
= p_1 \begin{bmatrix}
  v_{11} \\
  v_{12} \\
  \vdots \\
  v_{1n}
\end{bmatrix} \cos(\omega_1 t - \theta_1) + p_2 \begin{bmatrix}
  v_{21} \\
  v_{22} \\
  \vdots \\
  v_{2n}
\end{bmatrix} \cos(\omega_2 t - \theta_2) + \ldots + p_n \begin{bmatrix}
  v_{n1} \\
  v_{n2} \\
  \vdots \\
  v_{nn}
\end{bmatrix} \cos(\omega_n t - \theta_n)
\]
(3.40)

If the two-degree-of-freedom system discussed above is given arbitrary starting conditions, the resulting motion would be the sum of the two principal modes in some proportion and would look as shown in Fig.13.
Fig. 13. Response of the two-degree-of-freedom system when given arbitrary starting conditions.

3.3 Proportional Damping

The assumption that systems have no damping is only hypothetical, since all structures have internal damping. As there are several types of damping, viscous, hysteretic, coulomb, aerodynamic etc., it is generally difficult to ascertain which type of damping is represented in a particular structure. In fact a structure may have damping characteristics resulting from a combination of all the types. In many cases, however, the damping is small and certain simplifying assumptions can be made.

3.3.1 Viscous Damping

The equations of motion of the two degree of freedom system with damping, shown in Fig. 9, are given by Eq. (3.2)

\[
\begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
c_1 + c_2 & -c_2 \\
-c_2 & c_2 + c_3
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
k_1 + k_2 & -k_2 \\
-k_2 & k_2 + k_3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\]  

(3.41)

In short form they can be written as
\[
[m]\{\ddot{x}\}+[c]\{\dot{x}\}+[k]\{x\} = \{F\} 
\]

(3.42)

Two assumptions are taken for granted before attempting solution of these equations. Firstly, that the type of damping is viscous, and secondly that the distribution of damping is proportional. By proportional damping it is implied that the damping matrix \([c]\) is proportional to the stiffness matrix or the mass matrix, or to some linear combination of these two matrices. Mathematically it means that either

\[
[c] = \alpha [m] \quad \text{or} \quad [c] = \beta [k] \quad \text{or} \quad [c] = \alpha [m] + \beta [k] 
\]

(3.43)

where \(\alpha\) and \(\beta\) are constants.

Because of the assumption of proportional damping, the coordinate transformation using the modal matrix for the free undamped case which diagonalizes the mass and stiffness matrices, will also diagonalize the damping matrix. Thus the coupled equations of motion for a proportionally damped system can also be uncoupled to single degree of freedom systems as shown in the following.

Substituting the coordinate transformation of Eq. (3.24) into Eq. (3.42) we obtain

\[
[m][v]\{\ddot{p}\}+[c][v]\{\dot{p}\}+[k][v]\{p\} = \{F\} 
\]

(3.44)

Pre-multiplying Eq. (3.44) by the transpose of the modal matrix, i.e., \([v]^T\) we obtain

\[
\]

(3.45)

It was shown before in Eq. (3.29) and (3.30) that because of the orthogonal properties of the mode shapes the mass and stiffness matrices are diagonalized, i.e.

\[
[v]^T[m][v] = [^v m_v] 
\]

and

\[
[v]^T[k][v] = [^v k_v] 
\]

Because of proportional damping i.e.

\[
[c] = \alpha [m] + \beta [k] 
\]

we have

\[
[v]^T[c][v] = [v]^T[\alpha [m] + \beta [k]][v] 
\]
\[ = \alpha [v]^T [m][v] + \beta [v]^T [k][v] \]

i.e. \([v]^T [c][v] = \alpha \left[ \begin{array}{c} m_i \\ m_i \\ \vdots \end{array} \right] + \beta \left[ \begin{array}{c} c_i \\ \vdots \end{array} \right] = \left[ \begin{array}{c} c_i \\ \vdots \end{array} \right] \]

where \( \left[ \begin{array}{c} c_i \\ \vdots \end{array} \right] \) is a diagonal matrix.

Thus substitution into Eq. (3.45) gives

\[ \left[ \begin{array}{c} m_i \\ \vdots \end{array} \right] \{ \ddot{p} \} + \left[ \begin{array}{c} c_i \\ \vdots \end{array} \right] \{ \dot{p} \} + \left[ \begin{array}{c} k_i \\ \vdots \end{array} \right] \{ p \} = [v]^T \{ F \} \] (3.46)

Eq. (3.46) represents an uncoupled set of equations for damped single degree of freedom systems. The \( i^{th} \) equation is

\[ m_i \dddot{p}_i + c_i \dot{p}_i + k_i p_i = \{ v_i \}^T \{ F \} = f_i \] (3.47)

which represents the equation of motion of a single degree of freedom system.

Since \( k_i = \omega^2 m_i \), from Eq. (3.19), Eq. (3.47) can be written as

\[ \dddot{p}_i + 2\zeta_i \omega_i \dot{p}_i + \omega_i^2 p_i = \left( \begin{array}{c} v_i \end{array} \right)^T \{ F \} = \frac{f_i}{m_i} \] (3.48)

Where

\[ \zeta_i = \frac{c_i}{2\sqrt{k_i m_i}} \] (3.49)

The solution of a damped single degree of freedom system, as described by Eq. (3.48) has been discussed previously. Once the solution of Eq. (3.48) is obtained for all \( p_i \),
the solution in terms of the original coordinates \( \{x\} \) can be deduced by transforming back, i.e. substituting for \( p_i \), in Eq. (3.24).

It should be noted that if the damping matrix is proportional to the stiffness matrix, i.e., \( [c] = \alpha[k] \) then from Eq. (3.48) we see that

\[
\zeta_i \propto \frac{k_i}{\sqrt{k_i m_i}} \propto \omega_i
\]

which means that the higher frequency modes will have higher damping ratios.

### 3.3.2 Hysteretic Damping

Hysteretic or structural damping was discussed under single degree of freedom systems. It was shown, that in this case the damping force is proportional to the elastic force, but as energy is dissipated, the force is in phase with the velocity. Thus for simple harmonic motion the damping force is given by \( j\gamma kx \). For a multi-degree of freedom system, the equations of motion with hysteretic damping can be written as

\[
[m]\{\ddot{x}\} + j\gamma[k]\{x\} + [k]\{x\} = \{F\}
\]

Changing to Principal Coordinates as shown in the section above leads to

\[
[m^T]\{\ddot{p}\} + (1 + j\gamma)[k^T]\{p\} = [v]^T\{F\}
\]

Thus each equation is of the form

\[
m_i\ddot{p}_i + (1 + j\gamma)k_i p_i = \{v_i\}^T\{F\}
\]

i.e.

\[
\ddot{p}_i + (1 + j\gamma)\omega_i^2 p_i = \frac{\{v_i\}^T\{F\}}{m_i}
\]

If

\[
\{F\} = \{\bar{F}\} e^{j\omega t}
\]

then

\[
\{p\} = \{\bar{p}\} e^{j\omega t}
\]

Substituting in Eq. (3.52) we obtain

\[
-\omega_i^2 \ddot{p}_i + (1 + j\gamma)\omega_i^2 p_i = \frac{\{v_i\}^T\{\bar{F}\}}{m_i} = \frac{\bar{F}_i}{m_i}
\]
the solution of which has been discussed previously.

### 3.4 Non-proportional Damping

#### 3.4.1 State-Space Method

When the damping matrix is not proportional to the mass or the stiffness matrix, neither the modal matrix nor the weighted modal matrix will diagonalize the damping matrix. In this general case of damping, the coupled equations of motion have to be solved simultaneously, or they need to be uncoupled using the state-space method. By this method the set of $n$ second order differential equations are converted to an equivalent set of $2n$ first order differential equations, by assigning new variables (referred to as state variables) to each of the original variables and their derivatives. To illustrate the procedure, the equations of motion for the two degree of freedom system shown in Fig.9 are written as

\[
\begin{bmatrix}
    m_1 & 0 \\
    0 & m_2
\end{bmatrix}
\begin{bmatrix}
    \ddot{x}_1 \\
    \ddot{x}_2
\end{bmatrix} - \begin{bmatrix}
    m_1 & 0 \\
    0 & m_2
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

or in partitioned matrix form as

\[
\begin{bmatrix}
    m_1 & 0 & c_1 & c_2 \\
    0 & m_2 & -c_2 & c_1
\end{bmatrix}
\begin{bmatrix}
    \ddot{x}_1 \\
    \ddot{x}_2
\end{bmatrix} + \begin{bmatrix}
    -c_2 & c_1 + c_2 \\
    c_1 + c_2 & -c_2
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2
\end{bmatrix} + \begin{bmatrix}
    k_1 + k_2 & -k_2 \\
    -k_2 & k_1 + k_2
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\]

Substituting

\[
\begin{align*}
    x_1 &= z_1 \\
    \dot{x}_1 &= \ddot{z}_1 = z_3 \\
    \ddot{x}_1 &= \ddot{z}_3 \\
    x_2 &= z_2 \\
    \dot{x}_2 &= \ddot{z}_2 = z_4 \\
    \ddot{x}_2 &= \ddot{z}_4
\end{align*}
\]

we get
which can be abbreviated to

$$[A]\{\ddot{z}\} + [B]\{\dot{z}\} = \{Q\}$$

(3.56)

It can be seen that while the second order equations have been reduced to first order equations, the number of equations have been doubled.

The solution of above equations for free vibration reveals that damped natural modes do exist, however, they are not identical to the undamped natural modes. For the undamped modes, various parts of the structure move either in phase or 180° out-of-phase with each other. For the non-proportionally damped structures, there are phase differences between the various parts of the structure, which result in complex mode shapes. This difference is manifested by the fact that for undamped natural modes all points on the structure pass through their equilibrium positions simultaneously, which is not the case for the complex modes. Thus the undamped natural modes have well-defined nodal points or lines and appear as a standing wave, while for complex modes the nodal lines are not stationary.

3.4.2 Forced Normal Modes of Damped Systems

For an n-degree-of-freedom system with viscous damping, the equations of motion for steady state sinusoidal excitation can be written in its general form as

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{F\} \sin \omega t$$

(3.57)

where the system inertia, damping and stiffness matrices $[m]$, $[c]$ and $[k]$ respectively, and they are assumed to be real symmetric and positive definite. If the damping is hysteretic, the second term would be given by $1/\omega[d]\{\dot{x}\}$, where $[d]$ is the hysteretic damping matrix. In the general case damping would be non-proportional and thus the damping matrix cannot be diagonalized using the normal mode transformation. For an
arbitrary set of forces \( \{ F \} \) and excitation frequency \( \omega \) the solution of Eq. (3.57) is rather complicated.

Although the responses at each coordinate \( \{ x \} \) are harmonic with the excitation frequency, they are not all in phase with each other or with the excitation force. If, however, a system with \( n \)-degree-of-freedom is excited by \( n \) number of forces which are either 0° or 180° out-of-phase (often called monophase or coherently phased forces), then for a particular ratio of forces, the response at each of the coordinates will be in phase with each other and lag behind the force by a common angle \( \theta \) (called the characteristic phase lag). Thus we have to determine the conditions which will produce a solution of the form

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix} = \begin{bmatrix}
  X_1 \\
  X_2 \\
  \vdots \\
  X_n
\end{bmatrix} \sin(\omega t - \theta) = \{ v \} \sin(\omega t - \theta) \tag{3.58}
\]

For any given excitation frequency \( \omega \), there exist \( n \) solutions of the type given by Eq. (3.58), where each of the modes \( \{ v \} \) is associated with a definite phase \( \theta \) and a corresponding distribution of forces \( \{ \Gamma \} \), which is required for its excitation. The response under these conditions is called the “forced normal modes” of the damped system, since every point of the system moves in phase and passes through its equilibrium position simultaneously with respect to the other points.

Substituting Eq. (3.58) in Eq. (3.57) gives

\[
\sin(\omega t - \theta)\left[ [k] - \omega^2 [m] \right]\{ v \} + \omega \cos(\omega t - \theta)\left[ c \right]\{ v \} = \{ F \} \sin \omega t \tag{3.59}
\]

Expanding the \( \sin(\omega t - \theta) \) and the \( \cos(\omega t - \theta) \) terms and separating the \( \sin \omega t \) and \( \cos \omega t \) terms we obtain

\[
\cos \theta\left[ [k] - \omega^2 [m] \right]\{ v \} + \omega \sin \theta [c] \{ v \} = \{ F \} \tag{3.60}
\]
\[
\sin \theta \left[[k] - \omega^2 [m]\right] \{v\} - \omega \cos \theta [c] \{v\} = \{0\} \quad (3.61)
\]

These equations contain three unknowns \(\{F\}, \{v\}\) and \(\theta\) since \(\omega\) is given. If \(\cos \theta \neq 0\), Eq. (3.61) may be divided by \(\cos \theta\) to give

\[
\left[\tan \theta \left[[k] - \omega^2 [m]\right] - \omega [c]\right] \{v\} = \{0\} \quad (3.62)
\]

Eq. (3.62) has a non-trivial solution if the determinant

\[
\left|\tan \theta \left[[k] - \omega^2 [m]\right] - \omega [c]\right| = 0 \quad (3.63)
\]

It is evident that for a given there are \(n\) values of \(\tan \theta_i (i = 1, 2, \cdots n)\) corresponding to the \(n\) eigenvalues and for each \(\tan \theta_i\) there is a corresponding eigenvector \(\{v\}_i\) satisfying the equation

\[
\left[\tan \theta_i \left[[k] - \omega^2 [m]\right] - \omega [c]\right] \{v\}_i = \{0\} \quad (3.64)
\]

If Eq. (3.64) is premultiplied by the transpose \(\{v\}_i^T\) and rearranged, we obtain

\[
\tan \theta_i = \frac{\omega \{v\}_i^T [c] \{v\}_i}{\{v\}_i^T \left[[k] - \omega^2 [m]\right] \{v\}_i} \quad (3.65)
\]

From Eq. (3.65) it can be seen that each of the roots \(\tan \theta_i\) is a continuous function of \(\omega\). For low values of \(\omega\), \(\tan \theta_i\) is small, i.e., \(\theta_i\) is a small angle. As \(\omega\) increases and approaches \(\omega_i\), the undamped natural frequency, one of the roots \(\theta_i\) (which can be named \(\theta_i\)) approaches the value \(\pi / 2\). As \(\omega\) is increased above \(\omega_i\), the denominator of Eq. (3.65) gets negative and \(\theta_i(\omega)\) gets larger than \(\pi / 2\). When \(\omega\) tends to \(\infty\), \(\theta_i(\omega)\) tends to \(\pi\). In a similar manner the remaining roots \(\theta_i\) \((i = 2, 3, \cdots n)\) can be plotted as a function of frequency \(\omega\), where \(\theta_i (i = 2, 3, \cdots n)\) is equal to \(\pi / 2\) at the \(i\)th undamped natural frequency \(\omega_i\). Thus
\( \theta_k \) is that root which has the value \( \pi/2 \) at the undamped natural frequency \( \omega = \omega_k \).

Having examined the variation of eigenvalues \( \tan \theta_i \) as a function of frequency, the mode shapes can now be investigated. It can be seen from Eqs. (3.63) and (3.64) that at any one frequency the mode shapes depend only on the shape of the damping matrix and not on its intensity. If every element in the matrix \([c]\) is multiplied by a constant factor, then Eq. (3.63) shows that the roots \( \tan \theta_i \) will all be increased by the same ratio. Thus Eq. (3.64), which determines the mode shapes, will be multiplied throughout by the same factor and the mode shape \( \{v\}_i \) will be unchanged. Eq. (3.64) can be re-written as

\[
[k] - \omega^2 [m] - \frac{\omega [c]}{\tan \theta_i} \{v\}_i = \{0\}
\]

When \( \omega \) is equal to one of the undamped natural frequencies, say \( \omega = \omega_1 \), then one of the roots \( \theta \) is 90° as shown above. Thus Eq. (3.66) which determines the mode shape for this root becomes

\[
[k] - \omega_1^2 [m] \{v\}_i = \{0\}
\]

It can thus be seen, that when the frequency is equal to one of the undamped natural frequencies, the mode shape for one of the roots (which is equal to \( \pi/2 \)) is identical to the Principal or Normal mode shape.

Attention can now be paid to the force ratio that is required to excite any one mode \( \{v\}_i \) for the corresponding root \( \theta_i \) at any one frequency. The force ratio required can be calculated from Eq. (3.60) namely

\[
\cos \theta_i \left[ [k] - \omega^2 [m] \right] \{v\}_i + \omega \sin \theta_i [c] \{v\}_i = \{\Gamma\}_i
\]

In the special case when \( \omega = \omega_1 \) one of the undamped natural frequencies, one of the roots \( \theta_i = \theta_1 = 90^\circ \) and Eq. (3.68) reduces to

\[
\omega_1 [c] \{v\}_i = \{\Gamma\}_i
\]
To illustrate the concepts discussed above consider the same numerical example of the two degree of freedom system of Fig. 9, but with the values of damping added as shown in Fig. 16.

![Two degree of freedom system with non-proportional damping](image)

Thus the equations of motion according to Eq. (3.1) become

\[
\begin{bmatrix}
5 & 0 \\
0 & 10
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix} +
\begin{bmatrix}
5c & -1c \\
-1c & 8c
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} +
\begin{bmatrix}
4 & -2 \\
-2 & 6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\sin \omega t
\]

For a non-trivial solution the determinant of Eq. (3.63) must be equal to zero, i.e.

\[
\begin{vmatrix}
\tan \theta \left[4 & -2 \\
-2 & 6
\right] - \omega^2 \begin{bmatrix}
5 & 0 \\
0 & 10
\end{bmatrix} - \omega c \begin{bmatrix}
5 & -1 \\
-1 & 8
\end{bmatrix}
= 0
\]

i.e.,

\[
\begin{vmatrix}
\tan \theta \left(4 - 5\omega^2\right) - 5\omega c & \tan \theta (-2) + \omega c \\
\tan \theta (-2) + \omega c & \tan \theta \left(6 - 10\omega^2\right) - 8\omega c
\end{vmatrix} = 0
\]

i.e.,

\[
\left\{\tan \theta \left(4 - 5\omega^2\right) - 5\omega c\right\} \left\{\tan \theta \left(6 - 10\omega^2\right) - 8\omega c\right\} - \left\{\tan \theta (-2) + \omega c\right\}^2 = 0
\]

which reduces to

\[
\left(20 - 70\omega^2 + 50\omega^4\right)\tan^2 \theta - 2\left(29 - 45\omega^2\right)\omega c \tan \theta + 39\omega^2 c^2 = 0 \quad (3.70)
\]

The undamped natural frequencies of the system are given by letting \(c = 0\), i.e., by the equation

\[
\left(20 - 70\omega^2 + 50\omega^4\right) = 0
\]
They are found to be \( \omega_1 = \sqrt{2/5} \approx 0.63 \) rad/s and \( \omega_2 = 1 \) rad/s, which obviously should be the same as those calculated under section 3.1.

Dividing Eq. (3.70) by \( \tan^2 \theta \) and substituting \( \sigma = \frac{\omega c}{\tan \theta} \) we get

\[
39\sigma^2 - 2\left(29 - 45\omega^2\right)\sigma + \left(20 - 70\omega^2 + 50\omega^4\right) = 0
\]

The roots of this equation are given by

\[
\sigma = \frac{\omega c}{\tan \theta} = \frac{\left(29 - 45\omega^2\right) \pm \sqrt{\left(29 - 45\omega^2\right)^2 - 39\left(20 - 70\omega^2 + 50\omega^4\right)}}{39}
\]

(3.71)

When \( \omega \) is equal to one of the undamped natural frequencies \( \omega_1 \) or \( \omega_2 \) the equation reduces to

\[
\sigma = \frac{\omega c}{\tan \theta} = \frac{\left(29 - 45\omega^2\right) \pm \left(29 - 45\omega^2\right)}{39}
\]

Thus when \( \omega = \omega_1 \) we get

\[
\sigma = \frac{\omega_1 c}{\tan \theta} = \frac{\left(29 - 45\omega^2_1\right) \pm \left(29 - 45\omega^2_1\right)}{39}
\]

so that \( \theta_1 = 90^\circ \) for the negative sign and

\[
\theta_1 = \tan^{-1}\left(\frac{39\omega_1 c}{2\left(29 - 45\omega^2_1\right)}\right)
\]

for the positive sign.

Similarly when \( \omega = \omega_2 \) we get

\[
\sigma = \frac{\omega_2 c}{\tan \theta} = \frac{\left(29 - 45\omega^2_2\right) \pm \left(29 - 45\omega^2_2\right)}{39}
\]

so that \( \theta_2 = 90^\circ \) for the negative sign and
\[ \theta_1 = \tan^{-1}\left(\frac{39\omega_c c}{2(29 - 45\omega_c^2)}\right) \] for the positive sign.

The variation of \( \theta_1 \) and \( \theta_2 \) can be plotted as a function of frequency using Eq. (3.71). The curves are shown in Fig.17 for three values of damping \( c \), corresponding roughly to light, medium and heavy damping. The shape of the curves are seen to be similar to those of Fig.2(b).

![Fig. 17. Variation of the roots \( \theta_1 \) and \( \theta_2 \) as a function of frequency for three values of damping.](image)

The mode shapes are obtained by substituting numerical values for the matrices \([m]\), \([c]\) and \([k]\) in Eq. (3.64), i.e.,

\[
\begin{bmatrix}
\tan \theta_1 \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} - \omega_c^2 \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}
- \omega_c \begin{bmatrix} 5 & -1 \\ -1 & 8 \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

i.e.

\[
\begin{bmatrix}
\tan \theta_1 (4-5\omega_c^2) - 5\omega_c \\
\tan \theta_1 (-2) + \omega_c
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (i = 1, 2)
\]

Expanding the first equation we get

\[
\left\{\tan \theta_1 (4-5\omega_c^2) - 5\omega_c\right\} v_1 + \left\{\tan \theta_1 (-2) + \omega_c\right\} v_2 = 0
\]
Thus
\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \frac{2 \tan \theta_i - \omega_c}{\tan \theta_i \left( 4 - 5 \omega^2 \right) - 5 \omega_c} = \frac{2 - \omega_c / \tan \theta_i}{\tan \theta_i \left( 4 - 5 \omega^2 \right) - 5 \omega_c / \tan \theta_i}
\]

Substituting for \( \omega_c / \tan \theta \) from Eq. (3.71) we obtain

\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \frac{49 + 45 \omega^2 \pm \sqrt{61 + 120 \omega^2 + 75 \omega^4}}{11 + 30 \omega^2 \pm 5 \sqrt{61 + 120 \omega^2 + 75 \omega^4}}
\]

(3.72)

\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \begin{cases}
49 + 45 \omega^2 + \sqrt{61 + 120 \omega^2 + 75 \omega^4} \\
11 + 30 \omega^2 + 5 \sqrt{61 + 120 \omega^2 + 75 \omega^4}
\end{cases}
\]

Fig. 18. The characteristic phase lag modes as a function of frequency

The characteristic phase lag modes are plotted as a function of frequency using Eq. (3.72) in Fig. 18. The positive sign now corresponds to \( i = 1 \) for the first mode and the negative sign corresponds to \( i = 2 \) for the second mode. Note, in the special case when \( \omega \) is equal to the first undamped natural frequency \( \omega_1 = \sqrt{2/5} \) for \( i = 1 \) we obtain

\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \begin{cases}
49 + 45 \omega^2 + \sqrt{61 + 120 \omega^2 + 75 \omega^4} \\
11 + 30 \omega^2 + 5 \sqrt{61 + 120 \omega^2 + 75 \omega^4}
\end{cases}
\]

which is the same as the first principal mode shape \( \begin{bmatrix} v_1 \end{bmatrix}_1 \).
At \( \omega = \omega_1 = \sqrt{215} \) for \( i = 2 \) we obtain

\[
\{v\}_{\omega_1} = \frac{-56}{32}
\]

Similarly, when \( \omega \) is equal to the second undamped natural frequency \( \omega_2 = 1 \) for \( i = 1 \) we obtain

\[
\{v\}_{\omega_2} = \frac{110}{121}
\]

At \( \omega = \omega_2 = 1 \) for \( i = 2 \) we obtain

\[
\{v\}_{\omega_2} = \frac{-2}{1}
\]

which is the same as the second principal mode shape \( \{v\}_2 \).

The force ratio required to excite any one mode \( \{v\}_i \) for the corresponding root \( \tan \theta_i \) at any one frequency can be calculated from Eq. (3.68), i.e.,

\[
\cos \theta_i \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} - \omega^2 \begin{bmatrix} 5 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \omega \sin \theta_i \begin{bmatrix} 5 & -1 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\]

(3.73)

The force ratio for each root can be plotted as a function of frequency by substituting the value of \( \theta \) for each \( \omega \) and the corresponding mode shape. Fig.19 shows the force ratios required as a function of frequency for the two \( \theta \)s.
Fig. 19. Force ratios required to excite the two modes as a function of frequency.

It is, however, interesting to calculate the force ratios at the undamped natural frequencies. As shown previously, at any one undamped natural frequency, one of the roots \( \tan \theta = \infty \), i.e., one of the \( \theta = 90^\circ \), and for that root the mode shape obtained is the Principal Mode Shape. Thus Eq. (3.73) reduces to

\[
\omega c \begin{bmatrix} 5 & -1 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\]

which gives the force ratio required to excite the principal mode shape at the corresponding undamped natural frequency.

Substituting the first undamped natural frequency \( \omega_1 = \sqrt{2/5} \) and \( v_1 / v_2 = 1 \) we obtain

\[
\sqrt{2/5} c \begin{bmatrix} 5 & -1 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\]

i.e.,

\[
F_1 / F_2 = 4/7 = \{ \Gamma_1 \}
\]
Substituting the second undamped natural frequency \( \omega_2 = 1 \) and \( \nu_1 / \nu_2 = -2/1 \) we obtain

\[
\begin{bmatrix}
5 & -1 \\
-1 & 8 \\
\end{bmatrix} \begin{bmatrix}
-2 \\
1 \\
\end{bmatrix} = \begin{bmatrix}
F_1 \\
F_2 \\
\end{bmatrix}
\]

i.e.,

\[ F_1 / F_2 = -11/10 = \{ \Gamma_2 \} \]

Before concluding this section it is important to recapitulate the following points:

(1) For each frequency of excitation there are as many characteristic phase angles as there are number of degrees of freedom, corresponding to certain sets of forces.

(2) For each characteristic phase lag there is a corresponding mode shape which varies with frequency. At the undamped natural frequencies one of the mode shapes is identical to the corresponding undamped Principal mode.

(3) The mode shapes depend on the shape of the damping matrix and not on the intensity of damping.

(4) In each mode the responses at the coordinates are all in phase, but lag behind the excitation force by an angle \( \theta \). At the undamped natural frequency, \( \theta = 90^\circ \) for one of the modes which is the principal mode.

(5) Orthogonal properties of phase lag modes also exist.

The orthogonal properties of the principal modes of vibration were demonstrated previously in section 3.1.3. To derive analogous properties of forced modes. Eq.(3.62) can be written for the ith eigenvalue and eigenvector and premultiplied by the transpose of the jth eigenvector. The procedure is repeated with i and interchanged giving the following two equations,

\[
\tan \theta_i \{v\}_{j}^T [k] - \omega^2 \{m\} \{v\}_{j} - \omega \{v\}_{j}^T [c] \{v\}_{i} = 0 \quad (3.74)
\]

\[
\tan \theta_j \{v\}_{i}^T [k] - \omega^2 \{m\} \{v\}_{i} - \omega \{v\}_{i}^T [c] \{v\}_{j} = 0 \quad (3.75)
\]

Since \([m]\), \([c]\) and \([k]\) are symmetric matrices Eq.(3.75) can be transposed to obtain
\[
\tan \theta_j \{v_j^\ast\}^T \left[ [k] - \omega^2 [m] \right] \{v_j\} - \omega \{v_j^\ast\}^T [c] \{v_j\} = 0
\] (3.76)

Subtracting Eq.(3.76) from Eq.(3.74) we obtain the orthogonal properties as

\[
\{v_j^\ast\}^T \left[ [k] - \omega^2 [m] \right] \{v_j\} = 0
\] (3.77)

And

\[
\{v_j^\ast\}^T [c] \{v_j\} = 0
\] (3.78)

provided \( \tan \theta_i \neq \tan \theta_j \).

Combining Eqs.(3.77) and (3.78) with Eq.(3.60) we obtain the third relation

\[
\{v_j^\ast\} \{F_i\} = 0 = \{v_j^\ast\} \{F_j\}
\] (3.79)