Analysis of Defects in Viscoelastic Solids by a Transformed Boundary Element Method

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Abstract

Due to the inclusion of time as an independent variable to describe the mechanical behavior of viscoelastic materials, the available analytical solutions have been obtained only for a few simplified problems. To study the mechanical behavior of viscoelastic solids, the numerical approaches such as finite element method (FEM) and boundary element method (BEM) are normally needed. The main advantages of BEM are the reduction of the problem dimension by one and the exact satisfaction of certain boundary conditions for particular problems if their associated fundamental solutions are embedded in boundary element formulation. Through the use of correspondence principle, the viscoelastic solids can be effectively treated in Laplace domain. To take advantage of the available fundamental solutions for the defects in anisotropic elastic materials, in this paper we use the transformed BEM to treat the problems of viscoelastic solids containing defects such as holes, cracks, or inclusions. By using the subregion technique, the problems with simultaneous existence of multiple holes, cracks, and inclusions can also be treated. The main feature of this proposed method is that no meshes are needed along the boundary of defects and the boundary conditions are satisfied exactly, which means that the present approach should be more efficient and correct.

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1. Introduction

Viscoelastic materials exhibit a time and rate dependence that is completely absent in the elastic materials. To study the mechanical behavior of viscoelastic solids, the numerical approaches such as
finite element method (FEM) and boundary element method (BEM) are normally needed. In this paper, we consider the approach of BEM due to their advantage of dimension reduction. Generally, there are three different approaches to linear viscoelastic analysis by BEM. The first formulates a BEM in Laplace transform domain and obtain the solution in time domain by numerical inversion [1,2]. The second formulates a BEM directly in time domain [3-5]. Although the second approach looks more direct and efficient, but the lack of fundamental solutions in time domain restricts its applicability. To combine the advantages of the previous two approaches, a mixed BEM was proposed by Schanz [6], which can solve the problem in time domain but rely on the fundamental solutions in Laplace domain. In order to take advantage of the available fundamental solutions for the defects or interfaces in anisotropic elastic materials [7], the first approach, i.e., the transformed BEM to treat the problems of viscoelastic solids containing defects such as holes, cracks, inclusions, and/or interfaces has been done in our recent work [8]. In that study, several examples considering interfaces, holes, cracks, and/or inclusions have been illustrated and compared with the existing analytical or numerical solutions to show their correctness and efficiency. Here, we like to further extend the transformed BEM to more complicated problems such as the interface corners between two dissimilar viscoelastic materials, an elastic inclusion inside a viscoelastic material, and the simultaneous existence of inclusions, cracks, and holes in viscoelastic solids.

2. Extended Stroh formalism for linear anisotropic viscoelasticity

In a fixed rectangular coordinate system $x_i, i=1,2,3$, let $u_i$, $\sigma_{ij}$, and $\varepsilon_{ij}$ be, respectively, the displacement, stress and strain. The constitutive laws for linear anisotropic viscoelastic materials, the strain-displacement relations for small deformations, and the equilibrium equations for static loading conditions can be written as [9]

$$\begin{align*}
\sigma_{ij}(t) &= C_{ijkl}(t)\varepsilon_{kl}(0) + \int_0^t C_{ijkl}(t-r)\frac{\partial \varepsilon_{kl}(r)}{\partial r} dr, \\
\varepsilon_{ij}(t) &= \frac{1}{2}\left(u_{i,j}(t) + u_{j,i}(t)\right), \\
\sigma_{ij}(t) &= 0,
\end{align*}$$

(1)

where $i,j,k,l=1,2,3$, and the repeated indices imply summation; a comma stands for differentiation; $C_{ijkl}(t)$ are the relaxation functions. Taking the Laplace transform of (1) gives

$$\begin{align*}
\hat{\sigma}_{ij}(s) &= s\hat{C}_{ijkl}(s)\hat{\varepsilon}_{kl}(s), \\
\hat{\varepsilon}_{ij}(s) &= \frac{1}{2}\left[\hat{u}_{i,j}(s) + \hat{u}_{j,i}(s)\right], \\
\hat{\sigma}_{ij}(s) &= 0,
\end{align*}$$

(2)

where $s$ is the transform variable and the Laplace transform $\tilde{f}(s)$ of $f(t)$ is defined as

$$\tilde{f}(s) = \int_0^\infty f(t)e^{-st} dt.$$ 

(3)

Equations (2) are identical to the basic equations of linear anisotropic elasticity, which means that the viscoelastic solutions of the Laplace transform domain can be obtained directly from the solutions of the corresponding elastic problems with the replacement of $C_{ijkl}$ by $s\hat{C}_{ijkl}(s)$, if the boundary of a viscoelastic body is invariant with time. This statement is the so-called correspondence principle between linear elasticity and linear viscoelasticity [9,10] and is applicable to anisotropic viscoelastic materials.

By using the correspondence principle and Stroh formalism for two-dimensional linear anisotropic elasticity [7,11], the general solutions satisfying the 15 partial differential equations (2) can be written as

$$\begin{align*}
\tilde{u} &= 2\text{Re}\left[Af(z)\right], \\
\tilde{\phi} &= 2\text{Re}\left[Bf(z)\right],
\end{align*}$$

(4a)

where

$$\tilde{u} = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix}, \quad \tilde{\phi} = \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \end{pmatrix}, \quad f(z) = \begin{pmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \end{pmatrix}, \quad z_0 = x_1 + \mu x_2, \quad A = [a, a_1], \quad B = [b, b_2, b_3].$$

(4b)
and Re stands for the real part. $\mathbf{u}$ and $\mathbf{\phi}$ are the displacement and stress function vectors in Laplace transform domain, and $\mathbf{\phi}_i$, $i = 1, 2, 3$ are related to the stresses in Laplace transform domain by

$$
\mathbf{\sigma}_i = -\mathbf{\phi}_{i,2}, \quad \mathbf{\sigma}_{i,2} = \mathbf{\phi}_{i,1}.
$$

(5)

$f(z)$ is a function vector composed of three holomorphic complex functions $f_\alpha(z)$, $\alpha = 1, 2, 3$, which will be determined through the satisfaction of boundary conditions. $\mu_\alpha$ and $(a_\alpha, b_\alpha)$ are the material eigenvalues and eigenvectors, which can be determined by the standard material eigenrelations [7].

3. Transformed boundary element method

By using the correspondence principle, the boundary integral equations in Laplace domain of a linear viscoelastic body are equivalent to those of linear elastic body, i.e.,

$$
c_i(\xi)\bar{u}_i(\xi, s) = \int_\Gamma \left[ \bar{u}_i^*(\xi, x, s) \bar{t}_j(x, s) - \bar{t}_j^*(\xi, x, s)\bar{u}_i(x, s) \right] d\Gamma(x), \quad i, j = 1, 2, 3,
$$

(6)

where $\Gamma$ denotes the boundary of the elastic solid; $\bar{u}_i(x, s)$ and $\bar{t}_j(x, s)$ are the Laplace transform of the displacements and surface tractions along the boundaries; $c_i(\xi)$ are the free term coefficients dependent on the location of $\xi$; $\bar{u}_i^*(\xi, x, s)$ and $\bar{t}_j^*(\xi, x, s)$ are the fundamental solutions of the corresponding anisotropic elastic problem, which can be found in [7] for the problems of interfaces, holes, cracks and inclusions.

After getting the fundamental solutions from the corresponding elastic problems, the unknowns remained in the boundary integral equations (6) are $\bar{u}_i$ and $\bar{t}_j$ over the boundary $\Gamma$. In boundary element formulation, the boundary $\Gamma$ is approximated by a series of elements, and the points $x$, displacements $\bar{u}_i$, and tractions $\bar{t}_j$ on the boundary are approximated by the nodal points, nodal displacement and nodal traction through suitable interpolation functions. In this study, the linear variation within each element is assumed for the boundary points, and the displacements and tractions in Laplace transform domain. Thus, by following the standard procedure for boundary element formulation [12], the Laplace transform of the displacements and tractions of the boundary nodes can be obtained by solving a system of linear algebraic equations. Once all the Laplace transform of the displacements and tractions on the boundary are determined, the values of the internal stresses and displacements in Laplace domain can be calculated by using the boundary integral equations (6) again, where $c_i(\xi) = \delta_{ij}$. After getting the displacements $\bar{u}_i$ and stresses $\mathbf{\sigma}_i$ in Laplace transform domain through the transformed boundary elements, their associated solutions in real time domain can then be determined by numerical inversion of Laplace transform, such as Schapery method [13], and Stehfest method [14]. To treat the problems of anisotropic viscoelastic bodies containing multiple interfaces, holes, cracks, and/or inclusions, the subregion technique considering the compatibility and equilibrium conditions between the interfaces of subregions is employed in this study [8].

4. Numerical examples

The special feature of the BEM proposed in this paper is that no meshes are needed along the boundaries of interfaces, holes, cracks and inclusions, and the meshes along the remaining boundaries can be made by relatively rough discretization. To show this special feature, several examples considering defects of viscoelastic solids have been illustrated in our recent study [8]. Here, we like to explore more
complicated problems such as the interface corners between two dissimilar viscoelastic materials, an elastic inclusion inside a viscoelastic material, and the simultaneous existence of inclusions, cracks, and holes in viscoelastic solids.

**Example 1: Interface corners**

Consider an interface corner between two dissimilar isotropic viscoelastic materials (Figure 1(a)). Both of the materials can be characterized by a shear relaxation function $G(t)$ and a constant bulk modulus $\kappa$. The relaxation function is considered to have the following form

$$G(t) = G_\infty + (G_0 - G_\infty) e^{-t/\tau}$$

(7)

where $G_0$ and $G_\infty$ are, respectively, the shear moduli at the initial and final states, and parameter $\tau$ is the relaxation time that determines the rate of decay. In this example,

$G_0 = 5.81\text{GPa}, \quad G_\infty = 2.65\text{GPa}, \quad \tau = 10\text{sec}, \quad \kappa = 12.59\text{GPa}$ for material 1,

$G_0 = 1.31\text{GPa}, \quad G_\infty = 0.13\text{GPa}, \quad \tau = 5\text{sec}, \quad \kappa = 2.84\text{GPa}$ for material 2.

The variations of stresses $\sigma_{xy}$ with time and location are shown, respectively, in Figure 1(b) and 1(c). From these figures we see that the stresses calculated by the commercial finite element ANSYS (with 27326 elements) are discontinuous across the interface and will approach to the solutions calculated by the present BEM (with 297 elements). The CPU times of this example are around 6 minutes for present BEM and 40 minutes for ANSYS, which shows that the present BEM is not only accurate but also efficient.

**Example 2: An elliptical elastic inclusion**

Consider an elliptical elastic inclusion embedded in a viscoelastic material (Figure 2(a)). Both of the materials are considered to be anisotropic and are characterized by the constant Poisson’s ratios $\nu_{ij}$, and the following relaxation functions: Young’s modulus $E_i(t)$ and shear modulus $G_{ij}(t)$,

$$E_i(t) = E_i^0 + (E_i^0 - E_i^\infty) e^{-t/\tau} \quad G_{ij}(t) = G_{ij}^0 + (G_{ij}^0 - G_{ij}^\infty) e^{-t/\tau}, \quad i, j = 1, 2, 3,$$

(9a)

in which

$$E_1^0 = 134.45\text{GPa}, \quad E_2^0 = E_3^0 = 11.03\text{GPa}, \quad G_{12}^0 = G_{13}^0 = 5.84\text{GPa}, \quad G_{23}^0 = 2.98\text{GPa},$$

$$E_i^\infty = \lambda E_1^0, \quad G_{ij}^\infty = \lambda G_{ij}^0, \quad \nu_{12} = \nu_{13} = 0.301, \quad \nu_{23} = 0.49.$$  

(9b)
and
\[ \lambda = 0.1, \tau = 10 \text{ min, for viscoelastic matrix}, \]
\[ \lambda = 1, \tau = 10^6 \text{ min, for elastic inclusion}. \]  

Figures 2(b) and 2(c) show the hoop stresses \( \sigma_{\theta\theta} \) around the inclusion boundary at some specific times for two different rotation angles of inclusions, \( \alpha = 0^\circ \) and \( \alpha = 45^\circ \). From the special arrangement given in (9), we see that at initial state the properties of matrix and inclusion are the same, and the entire solid behaves like a homogeneous material. Thus, the solutions shown in Figures 2(b) and 2(c) can be checked by \( \sigma_{\theta\theta} \bigg|_{\psi = 0^\circ} = \sigma_{\theta\theta} \bigg|_{\psi = 45^\circ} \) for the initial state \( (t = 0) \). From these two figures, it is interesting to observe that the hoop stresses of certain points around the inclusion do not change in time.

**Example 3: One crack, one hole and one inclusion**

Consider the simultaneous existence of crack, hole and rigid inclusion in viscoelastic materials. The viscoelastic material is the same as material 1 of example 1.

The geometry and boundary conditions are shown in Figure 3(a) where \( \alpha = 1 \text{mm}, W = 2W = 40 \text{a} \) and \( \sigma_0 = 1 \text{MPa} \). The stress intensity factors \( K_i \) at points \( A \) and \( B \) versus the distances between crack and inclusion/hole are shown in Figures 3(b) and 3(c). The results show that \( K_i \) are increasing

Fig. 2. (a) An elliptical elastic inclusion in a viscoelastic material subjected to a uniform tension \( L/W = 1, a/W = 2h/W = 1/20 \); (b) The hoop stress \( \sigma_{\theta\theta} \) around the inclusion boundary \( (\alpha = 0^\circ) \); (c) The hoop stress \( \sigma_{\theta\theta} \) around the inclusion boundary \( (\alpha = 45^\circ) \).

Fig. 3. (a) A crack, hole and rigid inclusion in viscoelastic materials subjected to a uniform tension; (b) Stress intensity factor \( K_i \) versus distance \( d_1 \) for point A and B; (c) Stress intensity factor \( K_i \) versus distance \( d_2 \) for point A and B.
simultaneously when the hole is approaching to the crack, whereas $K_I^A$ and $K_I^B$ are decreasing when the rigid inclusion is approaching to the crack. In other words, the presence of hole/rigid inclusion will enhance/reduce the stress intensity of cracks. It is also observed that $K_I^A$ is always lower than $K_I^B$ at the same situation, and both of them will always increase when the time passes.

5. Conclusion

The transformed BEM developed in this paper are applicable for viscoelastic solids containing defects such as holes, cracks, or inclusions. The special feature of the present transformed BEM is that no meshes are needed along the boundaries of defects, and the meshes along the remaining boundaries can be made by relatively rough discretization. To show this special feature, several examples considering defects of viscoelastic solids are illustrated in this paper, such as the interface corners between two dissimilar viscoelastic materials, an elastic inclusion inside a viscoelastic material, and the simultaneous existence of inclusions, cracks, and holes in viscoelastic solids. The results show that the present transformed BEM is not only accurate but also efficient.

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