Fracture Analyses for Interface Corners in Elastic Materials Subjected to Thermal Loading

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Abstract. By employing the Stroh formalism for two-dimensional anisotropic thermoelasticity, fracture analyses of interface corners between two dissimilar anisotropic elastic materials under thermal loadings are considered in this paper. It was proved that the consideration of thermal effects will not influence the stress singularity but will induce heat flux singularity if the singularity of the temperature field is not permissible. To calculate the stress intensity factors via path independent H-integral, it was found that the one proposed previously for the mechanical loading conditions should be modified by adding an additional surface integral accounting for the thermal effects. Two examples considering cracks and corners in isotropic plates are presented to show the correctness and validity of the modified H-integral.

Introduction

Many engineering objects, e.g. electronic packages, engines of power vehicles, solar panels, and so on, are operated in thermal environments. It is known that the change in system temperature, the heat flux given on the object surface, and the heat generation inside the object can make the object deform, and hence the stresses will be induced when some restrictions on deformation are imposed, such as the clamped boundary condition or the perfect-bonded condition along the interface between dissimilar materials. Interface corners are the structures commonly appearing within the engineering objects and lots of failure initiate from these critical regions due to the discontinuities of geometry and material properties. Hence, fracture analyses for estimating the dangerous degree and the mode of fracture of interface corners in elastic materials subjected to thermal loading are needed urgently. Orders of stress singularity and stress intensity factors are two commonly used parameters when we perform fracture analyses within the category of linear elastic fracture mechanics (LEFM). This paper provides an accurate, efficient, and systematical solution technique to calculate these two parameters for interface corners between dissimilar elastic materials subjected to thermal loading.

To understand the mechanical behavior of anisotropic elastic materials under thermal environments, Stroh formalism [1,2] for two-dimensional anisotropic thermoelasticity has been employed in many works. By this formalism, the analytical closed-form solutions for the orders of heat flux/stress singularity and the near tip solutions of the multi-material anisotropic wedges under thermal loadings have been obtained [3]. To understand the fracture behavior of interface corners, a unified definition of stress intensity factors connecting cracks, corners, interface cracks, and interface corners was proposed in [4]. In that work, in order to avoid the oscillatory singular problems of interface corners a path-independent H-integral was suggested to calculate the stress intensity factors. Based on these works, in this paper we try to extend the study of interface corners from mechanical loading condition to the thermal loading condition.

Two-dimensional Anisotropic Thermoelasticity
In a fixed rectangular coordinate system \( x_i, i = 1, 2, 3 \), let \( u_i, \sigma_{ij}, \varepsilon_{ij}, T \) and \( h_i \) be, respectively, the displacement, stress, strain, temperature, and heat flux. If we ignore the coupling terms between the elastic deformation and heat conduction, the heat conduction, the energy equation, the strain-displacement relation for the small deformations, the constitutive law for the linear anisotropic elastic materials, and the equilibrium equations for the static loading conditions can be written as [5]

\[
\begin{align*}
    h_i &= -k_{ij}T_j, \\
    \varepsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}), \\
    \sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - \beta_{ij}T, \\
    \sigma_{ij,j} &= 0,
\end{align*}
\]

(1)

where repeated indices imply summation, a comma stands for differentiation, and \( C_{ijkl}, k_{ij} \) and \( \beta_{ij} \) are the elastic constants, heat conduction coefficients and thermal moduli, respectively. \( C_{ijkl} \) are assumed to be fully symmetric, i.e.,

\[
C_{ijkl} = C_{ijlk} = C_{klij} = C_{klij}
\]

and are required to be positive definite due to the positiveness of the strain energy. \( \beta_{ij} \) and \( k_{ij} \) are also assumed to be symmetric, i.e.,

\[
\beta_{ij} = \beta_{ji} \quad \text{and} \quad k_{ij} = k_{ji}.
\]

Equation 1 constitutes 19 partial differential equations in terms of three coordinate variables \( x_i, i = 1, 2, 3 \). If the deformations are considered to be dependent upon two coordinate variables \( x_1 \) and \( x_2 \) only, a general solution satisfying these 19 equations has been obtained as [6,7]

\[
\begin{align*}
    T &= 2 \text{Re}\{g'(z_i)\}, \\
    h &= -2 \text{Re}\{(k_i + r_k)g'(z_i)\}, \\
    u &= 2 \text{Re}\{(Af(z) + cg(z))\}, \\
    \phi &= 2 \text{Re}\{(Bf(z) + df(z))\},
\end{align*}
\]

(2a)

where

\[
\begin{align*}
    h &= \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}, \\
    u &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \\
    \phi &= \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix},
\end{align*}
\]

(2b)

and

\[
\begin{align*}
    k_1 &= \begin{bmatrix} k_{11} \\ k_{21} \\ k_{31} \end{bmatrix}, \\
    k_2 &= \begin{bmatrix} k_{12} \\ k_{22} \\ k_{32} \end{bmatrix}, \\
    f(z) &= \begin{bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \end{bmatrix}, \\
    A &= [a_1 a_2 a_3], \\
    B &= [b_1 b_2 b_3],
\end{align*}
\]

(2c)

and

\[
\begin{align*}
    z_k &= x_1 + \mu_k x_2, \\
    k &= 1, 2, 3, \\
    z_i &= x_1 + \alpha_2
\end{align*}
\]

(2d)

In the above, \( \text{Re} \) stands for the real part, \( u \) is the displacement vector and \( \phi \) is the stress function vector which is related to the stresses \( \sigma_{ij} \) and surface tractions \( t \) by

\[
\begin{align*}
    \sigma_{ij} &= -\phi_{i,2}, \\
    \sigma_{i2} &= \phi_{i,1}, \\
    t &= \partial \phi / \partial s,
\end{align*}
\]

(3)

where \( s \) is the arc length measured along the curved boundary. \( f(z) \) is a function vector composed of three holomorphic complex functions \( f_k(z_k) \) which will be determined through the satisfaction of the boundary conditions. The argument \( z_k \) of each component function \( f_k \) contains \( \mu_k \) which is the material eigenvalue that has been proved to have three pairs of complex conjugates [2]. Moreover, in Eq. 2 the first three material eigenvalues are arranged to be those having the positive imaginary part.
A and B defined in Eq. 2c are two $3 \times 3$ complex matrices of which $(a_k, b_k)$, $k=1, 2, 3$, are the material eigenvectors associated with the first three material eigenvalues $\mu_k$. Corresponding to $\mu_k$ and $(a_k, b_k)$, the material eigenvalues and eigenvectors associated with the thermal effects, $\tau$ and $(c, d)$, can also be proved to have one pair of complex conjugate. Like $\mu_k$, the first $\tau$ is arranged to be the one with positive imaginary part. These material eigenvalues and eigenvectors can be determined by the following eigen-relation [3,6,7]

$$N_\xi = \mu N_\xi, \quad N_\eta = N_\eta + \gamma,$$

(4a)

where

$$N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_1^T \end{bmatrix}, \quad \xi = \begin{bmatrix} a \\ b \end{bmatrix},$$

(4b)

$$\gamma = -\begin{bmatrix} 0 & N_2 \\ I & N_1^T \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \eta = \begin{bmatrix} c \\ d \end{bmatrix}, \quad \beta_1 = \begin{bmatrix} \beta_{11} \\ \beta_{21} \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} \beta_{12} \\ \beta_{22} \end{bmatrix},$$

and

$$N_1 = -T^{-1}R^T, \quad N_2 = T^{-1}N_2^T, \quad N_3 = RT^{-1}R^T + Q.$$  

(4c)

In the above the superscripts $T$ and $^{-1}$ stands for the transpose and inverse, respectively. The $3 \times 3$ real matrices $Q$, $R$ and $T$ are defined as

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}.$$  

(4d)

Note that to determine $\eta$ from Eq. 4a$_2$, one should first find $\tau$ from the following characteristic equation,

$$k_{22} \tau^2 + 2k_{12} \tau + k_{11} = 0.$$  

(5)

Interface Corners under Thermal Effects

Consider an interface corner between two dissimilar anisotropic elastic materials (Fig. 1). Assume perfect bond along the interface. The displacement, traction, temperature, and heat flux continuity across the interface $\theta = 0$ can be written as

$$u_1(0) = u_2(0), \quad \phi_1(0) = \phi_2(0), \quad T_1(0) = T_2(0), \quad h_1^*(0) = h_2^*(0),$$

(6a)

where the subscripts 1 and 2 denote the values related to materials 1 and 2, and the argument 0 denotes the values on the interface. The two outer surfaces of the interface corner are assumed to be traction free which can be expressed by stress function as

$$\phi(\theta_o) = \phi(\theta_o) = 0.$$  

(6b)

Four different thermal conditions on the two outer surfaces are considered in this paper. They are

- isothermal-isothermal: $T_1(\theta_o) = T_2(\theta_o) = 0$,
- insulated-insulated: $h_1^*(\theta_o) = h_2^*(\theta_o) = 0$,
- insulated-isothermal: $h_1^*(\theta_o) = T_2(\theta_o) = 0$,
- isothermal-insulated: $T_1(\theta_o) = h_2^*(\theta_o) = 0$.  

(6c)
In the above, $h^*$ is the heat flux in the direction normal to the interface. If the interface is located on a radial line with angle $\theta$ measured counterclockwise from the $x_1$-axis, the heat flux $h^*$ normal to the interface can be calculated from the vector coordinate transformation as

$$h^* = -h_1 \sin \theta + h_2 \cos \theta \quad (7a)$$

Substituting Eq. 2a into Eq. 7a and using the characteristic equation for $r$ given in Eq. 5, we get

$$h^* = ik_0 \delta(1-\delta)r^{-1-\delta}\{q_1 \bar{r}^{-\delta}(\theta) - q_2 \bar{r}^{-\delta}(\theta)\}, \quad (7b)$$

where $k_0$ is a real constant defined by

$$k_0 = -ik_{22}(\tau - \bar{\tau})/2 = \sqrt{k_{11}k_{22} - k_{12}^2} \quad (7c)$$

Fig. 1: An interface corner.

By using the general solutions given in Eq. 2, the near tip solutions satisfying boundary conditions Eq. 6 have been obtained in [3]. In that study, it concludes that the consideration of thermal effects will not influence the stress singularity but will induce heat flux singularity if the singularity of the temperature field is not permissible. Therefore, to consider the orders of stress singularity and their associated intensity factors, only the near tip solutions under mechanical loading (without consideration of the thermal effects) are employed in our following studies. They are [4]:

$$u(r, \theta) = r^{1-\delta_e} V(\theta) < r^{i\epsilon_{\alpha}} > c, \quad \phi(r, \theta) = r^{1-\delta_e} \Lambda(\theta) < r^{i\epsilon_{\alpha}} > c, \quad (8a)$$

where

$$V(\theta) = \begin{bmatrix} \eta_i(\theta) & \eta_2(\theta) & \eta_3(\theta) \end{bmatrix}, \quad \Lambda(\theta) = \begin{bmatrix} \lambda_i(\theta) & \lambda_2(\theta) & \lambda_3(\theta) \end{bmatrix}, \quad (8b)$$

and $\eta_i(\theta)$ and $\lambda_i(\theta)$, $i=1,2,3$, are the eigenfunctions [4,8]; $c$ is a coefficient vector related to the stress intensity factors $k$; $\delta = \delta^*_e + i\epsilon_{\alpha}$, $\alpha = 1,2,3$, is the singular order determined by

$$\|K^{(3)}_e\| = 0. \quad (9a)$$

$K^{(3)}_e$ is one of the submatrices of $K_e$ defined by

$$K_e = \begin{bmatrix} K^{(1)}_e & K^{(2)}_e \\ K^{(3)}_e & K^{(4)}_e \end{bmatrix}, \quad K_e = \hat{N}^{1-\delta}(\theta_2, \theta_1)\hat{N}^{1-\delta}_{\bar{\tau}}(\theta_0, \theta_1), \quad (9b)$$

and $\hat{N}$ is the key matrix [9] defined by

$$\hat{N}(\theta, \alpha) = \cos(\theta - \alpha)I + \sin(\theta - \alpha)N(\alpha), \quad (9c)$$
where $I$ is a $6 \times 6$ unit matrix and $N(\alpha)$ is the generalized fundamental matrix of $N$. It has been proved that the key matrix is related to the material eigenvalues $\mu_k$ and eigenvector matrices $A$ and $B$ by [9]

$$
\hat{N}^{-\delta}(\theta, \alpha) = \begin{bmatrix} A & \overline{A} \\ B & \overline{B} \end{bmatrix} \begin{bmatrix} \hat{\mu}_k^{-\delta}(\theta, \alpha) > \\ 0 \end{bmatrix},
$$

in which the overbar denotes the complex conjugate; the angular bracket <> stands for a diagonal matrix in which each component is varied according to the subscript $k$, e.g., $<z_k>$ = diag. $[z_1, z_2, z_3]$; $\hat{\mu}_k(\theta, \alpha)$ is related to the material eigenvalue $\mu_k$ by

$$
\hat{\mu}_k(\theta, \alpha) = \cos(\theta - \alpha) + \sin(\theta - \alpha)\mu_k(\alpha),
$$

$$
\mu_k(\alpha) = \frac{\mu_k \cos \alpha - \sin \alpha}{\mu_k \sin \alpha + \cos \alpha}, \quad k = 1, 2, 3.
$$

The singular order calculated from Eq. 9 may be integer (positive or negative or zero), real or complex. If only the stress singularity is concerned and the strain energy should be bounded, only the region $0 < \Re(\delta) < 1$ is considered in our study.

**Stress Intensity Factors**

A proper definition for the interface corner has been proposed recently [4], which can be reduced to the conventional definition for cracks in homogeneous anisotropic materials or for cracks along the interfaces between two dissimilar anisotropic materials. It is

$$
K_{II} = \lim_{r \to 0, \theta \to 0} \frac{1}{2\pi r} \Lambda < (r / \ell)^{-i\delta} \Lambda^{-1} \begin{bmatrix} \sigma_{r\theta} \\ \sigma_{\theta\theta} \\ \sigma_{\theta\theta} \end{bmatrix},
$$

or in matrix form

$$
k = \lim_{r \to 0} \frac{1}{2\pi r} \Lambda < (1 - \delta_r + ie_\alpha) \ell^{ie_\alpha} > \Lambda^{-1} \Phi(r, 0).
$$

In Eq. 10, $\ell$ is a length parameter which may be chosen arbitrarily as long as it is held fixed when specimens of a given material pair are compared; and $\Lambda = \Lambda(0)$. It has been shown that the stress intensity factors $k$ is related to the coefficient vector $c$ by [4]

$$
k = \sqrt{2\pi} \Lambda < (1 - \delta_r + ie_\alpha) \ell^{ie_\alpha} > c.
$$

**Path-independent H-integral for Thermoelastic Problems**

To provide a stable and efficient computing approach for the general mixed-mode stress intensity factors, a path-independent H-integral [10,11] has been proposed for either two-dimensional or three dimensional problems [4,8,12]. However, in those studies they consider only the pure mechanical loading conditions. For thermoelastic problems, by using the body force analogy [13], it can be proved that the following modified H-integral is path-independent [14-16]:

$$
H = \int_{\Gamma} (u^T \hat{t} - \hat{u}^T t) d\Gamma + \int_{S} \beta_y (\hat{T} \hat{e}_{y} - T e_{y}) dS,
$$

where $u$, $t$ and $T$ are the displacement vector, traction vector and temperature of the actual system, and $\hat{u}$, $\hat{t}$ and $\hat{T}$ are those of the complementary system; the path $\Gamma$ emanates from the lower wedge
flank $\theta_0$ and terminates on the upper wedge flank $\theta_2$ in counterclockwise direction and $S_r$ is the area enclosed by the path $\Gamma$. In our study, the complementary solutions are selected to be the one with the singular order $2-\delta$ and $\hat{T}=0$.

The difference between the modified H-integral Eq. 12 and the H-integral for pure mechanical loading problem is the additional area integral in the second term of Eq. 12 whose complementary strain $\hat{\epsilon}_{ij}$ has high singularity behavior as $r^{\delta-2}$. This term will cause tremendous numerical error, and hence should be treated with special attention \[16\]. By using the near tip solutions Eq. 8, it has been proved that the stress intensity factor $k$ is related to the H-integral by \[4\]

$$ k = \sqrt{2\pi} \Lambda < 1-\delta_r + i\sigma_a > \hat{H}^{-1} h, $$

where $H^+$ is a matrix related to the near tip and complementary near tip solutions and $h$ is a vector consisting of the value of H-integral calculated with specified complementary solution.

Numerical Examples

Firstly, to validate the proposed approach, a center crack in an isotropic square plate under pure mode I (or mode II) thermal loading will be analyzed to compare our numerical results with the solutions available in the literatures. Then, an example discussing the effect of corner angle on the orders of stress singularity and stress intensity factors will be investigated.

**Example 1: A center crack in an isotropic plate under mode I thermal loading.** This example had been done in \[17-19\]. In Fig. 2a, a crack is embedded in a square isotropic plate whose Young’s modulus, Poisson’s ratio, coefficient of thermal expansion, and heat conduction coefficient are $E=1\text{MPa}$, $\nu=0.3$, $\alpha=10^{-4}\text{C}^{-1}$, $k=1\text{W/m}^\circ\text{C}$, respectively. The half crack length $a=4\text{mm}$ and the half plate width $w=10a$. The crack surfaces are subjected to the constant temperature $T_{in}=0\text{C}$, while all the outer edges of the plate are maintained at the temperature $T_{out}=100\text{C}$. Table 1 shows the comparison of our results with those presented in the literature and reveals good agreement among our results and all the reference solutions. The path-independent property of H-integral is checked numerically via three circular paths with different radii, $r$.

**Example 2: A center crack in an isotropic plate under mode II thermal loading.** The analytical solutions of this example has been provided in \[7\]. Fig. 2b displays a center crack in an isotropic square plate whose Young’s modulus, Poisson’s ratio, coefficient of thermal expansion, and heat conduction coefficient are $E=218.4\text{GPa}$, $\nu=0.3$, $\alpha=16.7\times10^{-6}\text{C}^{-1}$, $k=1\text{W/m}^\circ\text{C}$, respectively. The half crack length $a=1\text{mm}$ and the ratio $w/a=30$ is used to simulate an infinite domain. The crack surfaces and both of the two vertical edges of the plate are all insulated (i.e. no heat flux across these edges). The top and bottom edges of this plate are, respectively, maintained at the constant temperatures $T_{up}=12.9938\text{C}$ and $T_{low}=-12.9938\text{C}$. Comparison between the present results and the reference solutions is also listed in Table 1. In this table, the data show that our results agree well with the analytical solution and the path-independent property of H-integral indeed holds.

**Example 3: An edge corner in an isotropic plate under thermal and mechanical loadings.** The geometrical data, boundary conditions, and loadings used in this example are shown in Fig. 3. Note that the corner surfaces are also under insulated condition which is not shown in Fig. 3. The material properties of this isotropic plate are the same as those of example 2. The mode II thermal loading used in example 2 and an extra mechanical tensile stress $\sigma_o=1\text{MPa}$ are applied on the two horizontal edges of the plate. The results of the first three orders of stress singularity for the edge corners ranging from $\varphi=0$ to $\varphi=80^\circ$ are shown in Fig. 4a, whereas Fig.4b shows the results of the stress intensity factors associated with the most critical singular order $\delta_c$. Without caring the difference of the units, from Fig. 4b we see that both of the absolute values of $K_i$ and $K_{II}$ increase monotonically with increasing $\varphi$. Also, the numerical data of $K_i$ and $K_{II}$ for the crack whose $\varphi=0^\circ$ agree with the
analytical solutions shown in the literature.

![Figure 2: A center crack in an isotropic plate under (a) mode I and (b) mode II thermal loadings.](image)

**Table 1: Comparison of stress intensity factors in examples 1 and 2.**

<table>
<thead>
<tr>
<th>r/a</th>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K_I$ [MPa×√mm]</td>
<td>$K_{II}$ [MPa×√mm]</td>
</tr>
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<td></td>
<td>present</td>
<td>[17]</td>
</tr>
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<td>0.01722</td>
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</table>

![Figure 3: An edge corner in an isotropic plate under thermal and mechanical loadings.](image)

Fig. 3: An edge corner in an isotropic plate under thermal and mechanical loadings.
Summary
An efficient approach to calculate the orders of stress singularity and stress intensity factors of interface corners between two dissimilar anisotropic elastic materials under thermal loading is proposed in this paper. A modified H-integral accounting for the thermal effects is suggested to calculate the stress intensity factors. Through the numerical examples shown in this paper, the correctness and validity of this modified H-integral are verified for the cases of cracks and corners.

References