BENDING OF AN ANISOTROPIC PLATE WEAKENED BY AN ELLIPTICAL HOLE

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SUMMARY: An anisotropic plate weakened by an elliptical hole has been studied extensively for two-dimensional problems. However, due to the mathematical difficulties, no analytical solution has been found for its corresponding plate bending problems. Based upon our recent development of Stroh-like formalism for bending theory of anisotropic plates, most of the relations for bending problems can be organized into the forms for two-dimensional problems. Thus, by using the Stroh-like formalism, it becomes easier to obtain the analytical solutions for problems of anisotropic plates containing an elliptical hole subjected to out-of-plane bending moments. The solutions for its associated crack problems can then be obtained by setting the minor axis of the ellipse approach to zero.

KEYWORDS: Anisotropic, Elliptical Hole, Plate Bending, Stroh Formalism

INTRODUCTION

Although the hole/crack problems are very important in engineering applications, most of the analytical solutions found in the literature are for two-dimensional problems. As far as the authors’ knowledge, the only related analytical solutions found in the literature are obtained by Lekhnitskii [1] and Lu and Mahrenholtz [2]. The former is obtained for the orthotropic plates weakened by a circular hole, which was derived nearly 65 years ago by using complex variable method. The latter is obtained for the general anisotropic plates containing a polygon-like hole, which was derived by the modified Stroh formalism. While the latter solution should cover the results presented by the former solution, no verification and comparison have been provided. Due to the complexity and non-verification of the solutions and the non-perfect eigen-relation provided by Lu and Mahrenholtz [2], it is necessary for us to find a simple, exact and general solution for this important problem.

Since the boundary conditions for the hole/crack problems are not easy to be satisfied by using the conventional methods of plate bending theory, most of the efforts are devoted in the complex variable methods. Unlike the progress of two-dimensional problems, no major advancement about the complex variable methods in plate bending theory has been developed during these few decades. Recently, owing to the efforts of several researchers, the complex variable methods in anisotropic elasticity have reached a big step by connecting Lekhnitskii formulation
and Stroh formalism [3, 4, 5, 6, 7, 8]. However, still very few contributions have been made to the plate bending problems. Through our experience in two-dimensional anisotropic elasticity, recently we develop a Stroh-like formalism for the bending theory of anisotropic plates [9]. In our theory, the deflections, the moments and the transverse shear forces can all be expressed in complex matrix form. Moreover, by careful re-organization, a Stroh-like compact and elegant solution form has been formulated. Through this re-organization, most of the relations look very like the Stroh formalism for two-dimensional linear anisotropic elasticity [10]. Hence, almost all the mathematical techniques developed for two-dimensional problems can lend to the plate bending problems. Borrowing from this analogy, a simple analytical solution for the bending of an anisotropic plate weakened by an elliptical hole is now obtained explicitly. Furthermore, by using this solution, its associated crack problems can also be studied by letting the minor axis of the elliptical hole to be zero.

**STROH-LIKE FORMALISM**

In the bending theory of anisotropic plates, the governing equation (combining the Kirchhoff plate assumptions, the equilibrium equations, the constitutive laws and the kinematic relations) can be expressed in terms of the lateral deflection as

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^2 \partial y^2} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} = q ,
\]

where \( q \) is the lateral load distribution, and \( w \) is the deflection of the mid-plane. The deflection \( w \) is then determined by solving this partial differential equation with the satisfaction of the boundary conditions set for the considered problems. After finding the deflection, all the other physical values such as the in-plane displacements, moments, transverse shear forces and internal stresses can all be obtained through the use of their relations with the deflection.

Although it seems not difficult to get a general solution for the deflection to satisfy the governing equation (1), it is really not easy to find a unique solution satisfying the boundary conditions for the complicated geometrical boundaries by using the conventional methods introduced in most of the texts of plates and shells. In order to solve the problems with hole/crack boundaries, a Stroh-like complex variable formalism for the bending theory of anisotropic plates [9] will be used in this paper. In this formalism, the general solutions satisfying the governing equation (1) can be expressed as follows.

\[
\alpha = \alpha_0 + 2 \text{Re}\{\mathbf{Aw}'(z)\}, \quad \phi = \phi_0 + 2 \text{Re}\{\mathbf{Bw}'(z)\},
\]

in which Re stands for the real part of the complex number; \( \alpha \) and \( \phi \) are, respectively, the slope and stress function vectors defined as

\[
\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} -\int M_y \, dx \\ -\int M_y \, dy \end{bmatrix},
\]

where \( M_y \) and \( M_y' \) are the bending moments. \( \alpha_0 \) and \( \phi_0 \) are the particular solutions related to the lateral load distribution \( q \). \( w'(z) \) is a function vector related to the differentiation of deflection and is defined as

\[
w'(z) = \begin{bmatrix} w_1'(z_1) \\ w_2'(z_2) \end{bmatrix}, \quad z_k = x + \mu_k y, \quad k = 1, 2.
\]
\( \mu_k, A \) and \( B \) are the material eigenvalues and eigenvector matrices, which can be determined from the following eigen-relation:

\[ N \xi = \mu \xi, \quad (3a) \]

where

\[ N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_T^T \end{bmatrix}, \quad \xi = \begin{bmatrix} a \\ b \end{bmatrix}, \quad (3b) \]

and

\[ N_1 = -T^{-1}R^T, \quad N_2 = T^{-1}N_2^T, \quad N_3 = RT^{-1}R^T - Q = N_3^T. \quad (3c) \]

In the above eigen-relation, the three \( 2 \times 2 \) real matrices \( Q, R \) and \( T \) are defined as

\[ Q = \begin{bmatrix} D_{22}^* & \frac{1}{2}D_{26}^* \\ -\frac{1}{2}D_{26}^* & \frac{1}{4}D_{66}^* \end{bmatrix}, \quad R = \begin{bmatrix} -\frac{1}{2}D_{26}^* & D_{12}^* \\ \frac{1}{4}D_{66}^* & -\frac{1}{2}D_{16}^* \end{bmatrix}, \quad T = \begin{bmatrix} \frac{1}{4}D_{66}^* & \frac{1}{2}D_{16}^* \\ \frac{1}{2}D_{16}^* & D_{11}^* \end{bmatrix}, \quad (4) \]

in which \( D_{ij}^* \) is the components of the inverse of the bending stiffness matrix \( D_{ij} \), i.e.,

\[ D_{ij}^* = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{12}^* & D_{22}^* & D_{26}^* \\ D_{16}^* & D_{26}^* & D_{66}^* \end{bmatrix} = D^{-1}. \quad (5) \]

It has been proved that the material eigenvalues \( \mu_k \) cannot be real due to the positive definiteness of strain energy and they will appear in two pairs of complex conjugates [11]. If the eigenvalues are assumed to be distinct and are arranged in the order that \( k_1, k_2 \) are those with positive imaginary parts, their associated eigenvectors will be independent each other and the eigenvector matrices \( A \) and \( B \) are defined as

\[ A = [a_1, a_2], \quad B = [b_1, b_2]. \quad (6) \]

For the convenience of readers’ reference, the explicit expressions of the fundamental matrix \( N \) and the eigenvectors \( a_k \) and \( b_k \) obtained in [9] are listed below.

\[ N_1 = \begin{bmatrix} -2D_{26} & 1 \\ \frac{D_{12}}{D_{22}} & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 4D_{66} - \frac{4D_{26}^2}{D_{22}} & -2D_{16} + \frac{2D_{12}D_{26}}{D_{22}} \\ -2D_{16} + \frac{2D_{12}D_{26}}{D_{22}} & D_{11} - \frac{D_{12}^2}{D_{22}} \end{bmatrix}, \quad N_3 = \begin{bmatrix} -\frac{1}{D_{22}} & 0 \\ 0 & 0 \end{bmatrix}, \quad (7) \]

and

\[ a_k = c_k \begin{bmatrix} h_k \\ g_k \end{bmatrix}, \quad b_k = c_k \begin{bmatrix} -\mu_k \\ 1 \end{bmatrix}, \quad (8a) \]

where

\[ g_k = \frac{D_{11}}{\mu_k} + D_{12}\mu_k + 2D_{16}, \quad h_k = D_{12} + D_{22}\mu_k^2 + 2D_{26}\mu_k, \quad (8b) \]

\[ c_k^2 = \frac{1}{2(g_k - \mu_k h_k)}, \quad k = 1, 2. \quad (8c) \]
ELLIPICAL HOLES

Consider an unbounded anisotropic plate weakened by an elliptical hole subjected to out-of-plane bending moments $M_x = M_1$, $M_y = M_2$, and $M_{xy} = 0$ at infinity (Figure 1). There is no load around the edge of the elliptical hole. The contour of the elliptical hole is represented by

$$y = a \cos \psi, \quad x = b \sin \psi,$$

where $2a$, $2b$ are the major and minor axes of the ellipse and $\psi$ is a real parameter. The boundary conditions of this problem can be expressed as

$$M_x = M_1, \quad M_y = M_2, \quad M_{xy} = 0 \quad \text{at infinity},$$

$$M_n = V_n = 0 \quad \text{along the hole boundary},$$

where the subscript $n$ denotes the direction normal to the boundary. $V_n$ is the effective transverse shear force defined as

$$V_n = Q_n + \frac{\partial M_{mn}}{\partial t},$$

where $Q_n$ and $M_n$, are respectively, the transverse shear force and twisting moment. Since the boundary considered in this problem is an elliptical boundary, it is not easy to find a solution satisfying (10b) due to the difficulty to describe $n$. The substitutes of the boundary conditions, which are suitable for the Stroh-like formalism, have been discussed in [9]. From the discussions of [9], we know that the boundary condition (10) can be expressed in terms of the stress function as

$$\phi = -M_1 y i_x - M_2 y i_y \quad \text{at infinity},$$

$$\phi = 0 \quad \text{along the hole boundary},$$

where

Fig. 1. An anisotropic plate weakened by an elliptical hole subjected to out-of-plane bending moments $M_1$ and $M_2$. 
\[ \mathbf{i}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{i}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]  

(12c)

Since there is no lateral load applied on the plate, i.e., \( q = 0 \), the particular solutions \( \phi_0 \) and \( \alpha_0 \) of (2a) are set to be zero. In order to satisfy the boundary condition at infinity, we add the stress function vector \( \phi^\infty \) and the slope vector \( \alpha^\infty \) to the general solution (2a). Thus, the solutions to this problem can be expressed as

\[
\phi = \phi^\infty + 2 \text{Re} \{ A w'(z) \}, \quad \alpha = \alpha^\infty + 2 \text{Re} \{ B w'(z) \},
\]

(13a)

where \( \phi^\infty \) and \( \alpha^\infty \) are [12]

\[
\phi^\infty = -M_1 y_i - M_2 x_i, \quad \alpha^\infty = M_1 (d_{1y} - d_{1x}) + M_2 (d_{4y} - d_{3x}),
\]

(14a)

and

\[
\mathbf{d}_1 = \begin{bmatrix} -D_{16}^2 / 2 \\ D_{14}^2 \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} D_{14}^2 \\ -D_{16}^2 / 2 \end{bmatrix}, \quad \mathbf{d}_3 = \begin{bmatrix} D_{18}^2 \\ -D_{16}^2 / 2 \end{bmatrix}, \quad \mathbf{d}_4 = \begin{bmatrix} D_{18}^2 \\ -D_{16}^2 / 2 \end{bmatrix}.
\]

(14b)

In order to satisfy the free edge condition around the hole boundary set in (12b), by referring to the solutions of the corresponding two-dimensional problems [13, 14] we select

\[
w'(z) = \zeta^{-1}_k c, \quad \zeta_k = \frac{z_k + \sqrt{z_k^2 - a^2 - \mu_k b^2}}{a - i\mu_k b},
\]

(15)

where the angular bracket \( <> \) stands for the diagonal matrix, i.e., \( \langle \zeta_k^{-1} \rangle = \text{diag} [\zeta_1^{-1}, \zeta_2^{-1}, \zeta_3^{-1}] \), and \( c \) is the unknown coefficient to be determined through the satisfaction of the boundary condition. By careful mathematical derivation with the assistance of the eigen-relation (3) and orthogonality relation [9], the unknown coefficient can be obtained in a simple form and the function vector \( w'(z) \) can be expressed as [12]

\[
w'(z) = \frac{1}{2} \langle \zeta_k^{-1} \rangle A^{-1} (iM_1 b_i + M_2 a_i).
\]

(16)

By combining the results shown in (13)-(16), the explicit solutions for the hole problems can now be expressed as

\[
\phi = \phi^\infty - M_1 b \text{Im} \{ A \langle \zeta_k^{-1} \rangle A^{-1} \} i_2 + M_2 a \text{Re} \{ A \langle \zeta_k^{-1} \rangle A^{-1} \} i_1,
\]

(17a)

\[
\alpha = \alpha^\infty - M_1 b \text{Im} \{ B \langle \zeta_k^{-1} \rangle A^{-1} \} i_2 + M_2 a \text{Re} \{ B \langle \zeta_k^{-1} \rangle A^{-1} \} i_1,
\]

(17b)

where \( \text{Im} \) stands for the imaginary part of a complex number.

With the solutions given in (17), all the physical values such as the deflection, in-plane displacements, moments, transverse shear forces and internal stresses in the whole field of the plate can all be obtained through the use of their relations with the slope vector and stress function vector [9]. In practical applications, one is usually interested in the moments around the hole boundary since most of the critical stress occurs along the hole boundary. Through the use of a special technique developed for the calculation of moments in the general curvilinear coordinates [12], we obtain
\[ M = -\frac{1}{\rho} \text{Im}\{ \frac{e^{-iy}}{h_1 g_2 - h_2 g_1} \frac{1}{D_1} \left[ b_0 \sin^3 \theta + b_1 \sin \theta \cos^2 \theta + b_2 \sin^2 \theta \cos \theta + b_3 \cos^3 \theta \right] \}, \]

\[ M_f = M_1 + M_2 + \frac{1}{\rho} \text{Im}\{ \frac{e^{-iy}}{h_1 g_2 - h_2 g_1} \frac{1}{D_2} \left[ b_4 \sin^3 \theta + b_5 \sin \theta \cos^2 \theta + b_6 \sin^2 \theta \cos \theta + b_7 \cos^3 \theta \right] \}, \]

where \( h_k \) and \( g_k \) are given in (8b), and

\[ \rho = \sqrt{a^2 \sin^2 \psi + b^2 \cos^2 \psi}, \]

\[ D_\mu = \sin^4 \theta - (\mu_1^2 + \mu_2^2) \sin^2 \theta \cos^2 \theta + \mu_1^2 \mu_2^2 \cos^4 \theta, \]

where the angle \( \theta \) is related to the parameter \( \psi \) by

\[ \rho \cos \theta = b \cos \psi, \quad \rho \sin \theta = a \sin \psi. \]

The coefficients \( b_0, b_1, \ldots, b_7 \) used in (18) are defined as

\[ \begin{align*}
 b_0 &= m_1 + m_2, \quad b_1 = -(m_1 \mu_1^2 + m_2 \mu_1^2), \quad b_2 = m_1 \mu_1 + m_2 \mu_2, \quad b_3 = -\mu_1 \mu_2 (m_1 \mu_2 + m_2 \mu_1), \\
 b_4 &= m_3 + m_4, \quad b_5 = -(m_3 \mu_1^2 + m_4 \mu_1^2), \quad b_6 = m_3 \mu_1 + m_4 \mu_2, \quad b_7 = -\mu_1 \mu_2 (m_3 \mu_2 + m_4 \mu_1),
\end{align*} \]

where

\[ \begin{align*}
 m_1 &= s_1 (g_2 \mu_2 a - ih_2 M_1 b), \quad m_2 = s_2 (-g_1 \mu_2 a + ih_1 M_1 b), \\
 m_3 &= (\mu_1 g_1 + h_1)(g_2 \mu_2 a - ih_2 M_1 b), \quad m_4 = (\mu_2 g_2 + h_2)(-g_1 \mu_2 a + ih_1 M_1 b),
\end{align*} \]

and

\[ s_k = \frac{D_{11}}{\mu_1} + 3D_{16} + (D_{12} + 2D_{66}) \mu_k + D_{26} \mu_k^2. \]

**CRACKS**

An elliptic opening can be made into a crack of length \( 2a \) by letting \( b \) to zero. The explicit solution (17) is then applicable to crack problems with \( b = 0 \). Thus, the solution to crack problems can be expressed as

\[ \phi = \phi^* + M_2 a \text{Re} \{ A(\zeta_2^{-1}) A^{-1} \} i_1, \]

\[ \alpha = \alpha^* + M_2 a \text{Re} \{ B(\zeta_2^{-1}) B^{-1} \} j_1, \]

where

\[ \phi^* = -M_2 x_1, \quad \alpha^* = -(M_1 d_1 + M_2 d_3) x. \]

**EXAMPLES**

**Example 1:**

Consider an orthotropic plywood plate weakened by a circular hole with radius \( a \). The thickness of the plate is \( h = 2.289 \text{mm} \). The material properties of the plywood are

\[ E_1 = 1.686 \text{GPa}, \quad E_2 = 0.139 \text{GPa}, \quad G = 0.07 \text{GPa}, \quad \nu_{12} = 0.31. \]

The bending stiffnesses calculated by using the materials properties and the plate thickness are

\[ D_{11} = 1.70, \quad D_{22} = 0.14, \quad D_{12} = 0.043, \quad D_{66} = 0.07 \text{ (GPa-mm)}^3. \]

With the eigen-relation (3), the material eigenvalues can be obtained as

\[ \mu_1 = 1.04 + 1.55i, \quad \mu_2 = -1.04 + 1.55i. \]
Substituting these data into (18), the moments around the circular hole subjected to out-of-plane bending moments $M_x = M_y = M_{xy} = 0$ at infinity are shown in Table 1, which are almost the same as those presented by Lekhnitskii [1]. Figure 2 shows the plot of moments around the hole boundary when the plate is subjected to $M_x = M_y = M_y = M_{xy} = 0$ at infinity.

Table 1 The moments around the circular hole of an orthotropic plywood plate subjected to out-of-plane bending moments $M_x = M_y = M_y = M_{xy} = 0$.

<table>
<thead>
<tr>
<th>Moments</th>
<th>$M_{nt}/M$</th>
<th>$M_y/M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle ($\psi$)</td>
<td>Present</td>
<td>Lekhnitskii [1]</td>
</tr>
<tr>
<td>0°</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15°</td>
<td>-0.0266</td>
<td>-0.0249</td>
</tr>
<tr>
<td>30°</td>
<td>-0.085</td>
<td>-0.0817</td>
</tr>
<tr>
<td>45°</td>
<td>-0.2369</td>
<td>-0.2326</td>
</tr>
<tr>
<td>60°</td>
<td>-0.5453</td>
<td>-0.5432</td>
</tr>
<tr>
<td>75°</td>
<td>-0.5948</td>
<td>-0.5960</td>
</tr>
<tr>
<td>90°</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Fig. 2 An orthotropic plywood plate weakened by a circular hole subjected to out-of-plane bending moments $M_x = M_y = M_y = M_{xy} = 0$. 
Example 2:

Consider an orthotropic plywood plate weakened by an elliptical hole with semi-axes $a = 2b$ subjected to out-of-plane bending moments $M_x = M_y = M$, $M_{xy} = 0$ at infinity. The material properties of the plywood are the same as those of Example 1. Figure 3 shows the results of the moments around the elliptical hole boundary, from which we see that the maximum value of $M_{nt}$ locates at the point $\psi = 30^\circ$ and equals to $1.37M$. The maximum value of $M_t$ locates at the point $\psi = 90^\circ$ and equals to $4.06M$.

![An orthotropic plywood plate weakened by an elliptical hole](image)

Fig. 3 An orthotropic plywood plate weakened by an elliptical hole with semi-axes $a = 2b$ subjected to out-of-plane bending moments $M_x = M_y = M$, $M_{xy} = 0$.

CONCLUSION

The explicit closed form solutions for an anisotropic plate containing an elliptical hole subjected to out-of-plane bending moments are obtained in this paper by applying the Stroh-like formalism [9]. The solutions for the crack problems are also obtained by setting the minor axis of the ellipse approach to zero. To verify their correctness and to show their generality, two examples are presented. One is an orthotropic plate weakened by a circular hole, the other is an orthotropic plate weakened by an elliptical hole. Since the analytical solutions existing in the literature are valid only for the orthotropic plates containing a circular hole. The comparison can only be made for this special case. Moreover, due to the mathematical difficulties when comparing the solutions analytically, only numerical comparison is presented in this paper. The comparison shows that our solutions are almost the same as those obtained in the literature for this special case. Beyond this special case, our solutions also cover general anisotropic materials with elliptical holes, which are not included in the Lekhnitskii’s solutions since they only consider orthotropic materials not general anisotropic materials and only circular holes not general elliptical holes which may also include cracks.

REFERENCES


