Advanced Fluid Mechanics Midterm Dec. 1, 2010

1. (a) Write down the definition of an incompressible flow? And explain its physical meaning. (4%)
(b) Explain the physical meaning of $\nabla \times \vec{V}$ where $\vec{V}$ is the velocity of a fluid motion described in Eulerian view $\vec{V}(x,y,z,t)$. (4%)
(c) An observer standing on a boat which travels with a velocity $\vec{V}_b = u_b \vec{i} + v_b \vec{j}$ across a river flowing at the velocity $\vec{V} = u \vec{i} + v \vec{j}$. What is the time rate of change of the vorticity the observer see for the river? (4%)
(d) Explain the relation between the circulation $\Gamma$ and vorticity $\vec{\omega}$. (4%)
(e) Write down the equation for Kelvin theorem and explain the physical meaning of Kelvin theorem (Helmholtz theorem of vorticity part II). (4%)

2. Derive the continuity equation from mass conservation principles using an infinitesimal control volume of rectangular shape and having dimensions with dimensions (dx, dy, dz), as shown in Fig. 1. Identify the net mass flow rate through each surface of this element as well as the rate at which the mass of the element is increasing. The resulting equation should be expressed in terms of the cartesian coordinates (x, y, z, t), the cartesian velocity components (u, v, w) and the fluid density $\rho$. (20%)

3. Say that we have a fluid of variable density at rest and acted upon by a conservative body force $f_j$ (per unit mass) so $f_j = \frac{\partial \phi}{\partial x_j} = \nabla \phi$. Derive the necessary relation between $\rho$ and $\phi$ such that fluid is indeed at rest and no motion at all.
Hint: use the momentum equation $\rho \left( \frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} \right) = -\frac{\partial p}{\partial x_j} + \mu \left( \frac{\partial^2 u_j}{\partial x_i^2} \right) + \rho f_j$ with $u_j=0$. 

![Fig.1](image-url)
4. Consider the two-dimensional flow field defined by the following velocity components:

\[ u = \frac{v}{1+t}, \quad v = 1, \quad w = 0 \]

For this flow field find the equation of:
(a) The streamline through the point (1,1) at t=0
(b) The pathline for a particle released at the point (1,1) at t=0

5. Assume a Newtonian fluid with \( \rho = \text{constant} \). Evaluate viscous dissipation \( \Phi \) for a laminar, isothermal Couette flow with \( u = u(y) = \frac{U y}{a} \) in the x direction.

\[ \Phi = \lambda \left( \frac{\partial u_i}{\partial x_i} \right)^2 + \mu \left( \frac{\partial u_i}{\partial x_j} \right) \frac{\partial u_j}{\partial x_i} \]

6. Consider the two-dimensional fluid flow with velocity components:

\[ u = ax + bx^2 + cy ; \quad v = dy + exy \]

where a, b, and c, are constants greater than zero.

(a) Assume density is constant and the mass is conserved, what can be said about constant d and e. (5%)
(b) Determine if the vorticity is zero somewhere in the flow field. Does this imply that the flow is irrotational? (5%)
(c) Find the rate of shear of this flow field? (5%)
(d) Find the rate of rotation of this flow field? (5%)

Hint:
\[ e_{ij} = \frac{\partial u_i}{\partial x_j} \]
**Answer 1**

(a) \( \frac{D\rho}{Dt} = 0 \): As we followed a flowing fluid element, the density of the fluid element does not change with time and with position.

(b) \( \nabla \times \vec{V} = \vec{\zeta} \): which is a measurement of the rotation of a fluid element as it moves in the flow.

(c) \( \frac{D\vec{\zeta}}{Dt} = \frac{\partial \vec{\zeta}}{\partial t} + u \frac{\partial \vec{\zeta}}{\partial x} + v \frac{\partial \vec{\zeta}}{\partial y} \)

(d) \( \Gamma = \oint_C \vec{V} \cdot d\vec{l} = \int_A \nabla \times \vec{V} \cdot d\vec{A} \); \( \Gamma \) is the measurement of the vorticity inside the area \( A \) which is enclosed by \( C \).

(e) \( \frac{D\Gamma}{Dt} = 0 \): If we follow the material loop as it flow, the circulation of the loop will not change.

Assumption:

- Inviscid fluid.
- Conservative body force
- Fluid density is constant or barotropic
由质量守恒
\[ \frac{\partial}{\partial t} \int_\Omega \rho \, dV + \oint_{\partial \Omega} \rho \mathbf{v} \cdot d\mathbf{a} = 0. \]

\[ \frac{\partial}{\partial t} \int_\Omega \rho \, dV = \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) \]

\[ \int_{\partial \Omega} \rho \mathbf{v} \cdot d\mathbf{a} = \int_{\partial \Omega} \left( \rho u \mathbf{i} + \rho v \mathbf{j} + \rho w \mathbf{k} \right) \cdot d\mathbf{a} \]

\[ \int_{\partial \Omega} \rho \mathbf{v} \cdot d\mathbf{a} = - \int_{\partial \Omega} \left( \rho \frac{\partial u}{\partial x} \mathbf{i} + \rho \frac{\partial v}{\partial y} \mathbf{j} + \rho \frac{\partial w}{\partial z} \mathbf{k} \right) \cdot d\mathbf{a} \]

\[ \int_{\partial \Omega} \rho \mathbf{v} \cdot d\mathbf{a} = \int_{\partial \Omega} \left( \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial w}{\partial z} \right) \, dV = \int_{\partial \Omega} \left( \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) \right) \, dV \]

\[ \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = 0. \]
Using the momentum equation:

\[ f \frac{\partial u_k}{\partial t} + f \cdot u_k \cdot \frac{\partial u_k}{\partial x_k} = -\frac{\partial p}{\partial x_k} + \frac{1}{\rho} \left( \frac{\partial (\rho u_k)}{\partial x_k} \right) + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right) \right] + \rho f_k. \]

If fluid is at rest \( \rightarrow u_k = 0 \):

\[ 0 = -\frac{\partial p}{\partial x_k} + \rho f_k. \]

\[ \Rightarrow \frac{\partial p}{\partial x_k} = \rho f_k. \]

\[ F_k = \frac{\partial y}{\partial x_k} = \nabla \psi = f_k. \]

\[ \nabla \psi = \rho \nabla \psi. \]

\[ \Rightarrow \nabla \psi = -\rho \nabla \psi. \]

\[ \nabla \left( \nabla \psi - \rho \nabla \psi \right) = 0. \]

\[ \nabla \left( \rho \nabla \psi \right) = 0. \]

\[ \nabla \psi \times \nabla \psi = 0. \]
Answer 4

Consider the two-dimensional flow field defined by the following velocity components:

\[ u = \frac{v}{1 + t}, \quad v = 1, \quad w = 0 \]

For this flow field find the equation of:

(a) The streamline through the point (1,1) at t=0

\[ \frac{dx}{ds} = u = \frac{v}{1 + t} \]

\[ \frac{dy}{ds} = v = 1 \]

Integration:

\[ \frac{dx}{ds} = \frac{1}{1 + t} \]

\[ x = \frac{s}{1 + t} + c_1 \]

at \( s=0, x=1, y=1 \)

\[ c_1 = 1 \]

\[ x = \frac{s}{1 + t} + 1 \]

\[ y = s + c_2 \]

at \( t=0, \)

\[ x = \frac{s}{1 + 0} + 1 = s + 1 \]

\[ y = s + 1 \]

The streamline is \( x = y \)

(b) The pathline for a particle released at the point (1,1) at t=0

\[ \frac{dx}{dt} = u = \frac{v}{1 + t} = \frac{1}{1 + t} \]

\[ \frac{dy}{dt} = v = 1 \]

\[ x = \ln(1 + t) + c_1 \]

\[ y = t + c_2 \]

at \( t=0, x=1, y=1 \)

\[ 1 = \ln(1 + 0) + c_1 \]

\[ c_1 = 1 \]

\[ c_2 = 1 \]

\[ x = \ln(1 + t) + 1 \]

\[ y = t + 1 \]

The pathline is \( y = e^{x-1} \)
\[ \Phi = \lambda \left( \frac{\partial u_k}{\partial x_k} \right)^2 + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} \]

\[ = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]

\[ = \mu \left( \frac{\partial u}{\partial y} \right)^2 \]

Where \( u = u(y) = \frac{U_y}{a} \)

\[ \frac{\partial u}{\partial y} = \frac{U}{a} \]
Answer 6

(a)
\[ \nabla \cdot \mathbf{V} = 0 \]
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]
\[ \rightarrow a + 2bx + d + ex = 0 \]
\[ \Rightarrow \begin{cases} a + d = 0 \\ 2b + e = 0 \end{cases} \]
\[ \Rightarrow \begin{cases} d = -a \\ e = -2b \end{cases} \]

(b)
(1)
\[ \nabla \times \mathbf{V} = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathbf{k} \]
\[ = (ey - c) \mathbf{k} \]

At \( y = \frac{c}{e} \), the vorticity is zero.

(2) No, this does not imply that the flow is irrotational.

(c) & (d)
\[ e_y = \frac{\partial u}{\partial x_j} \]
\[ = \frac{1}{2} \left( \frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial u}{\partial x_j} - \frac{\partial u}{\partial x_i} \right) \]

Rate of shear
\[ = \frac{1}{2} \left( \frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial x_i} \right) \]
\[ = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]
\[ = \frac{1}{2} (c + ey) \]
Answer 6
(c) & (d)

Rate of rotation
\[ \frac{1}{2} \left( \frac{\partial u_i}{\partial x} - \frac{\partial u_j}{\partial x_i} \right) \]
\[ \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \]
\[ \frac{1}{2} (c - ey) \]
or

Rate of rotation
\[ \frac{1}{2} \left( \frac{\partial u_i}{\partial x} - \frac{\partial u_j}{\partial x_i} \right) \]
\[ \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \]
\[ \frac{1}{2} (ey - c) \]