Advanced Fluid Mechanics: Homework #4

1. Please read Ch9.1~Ch9.2 in IG Currie Textbook and derive boundary-layer equations. Prove the $y$-direction momentum equation $\frac{\partial p}{\partial y} = 0$ for and boundary layer thickness $\delta$:

$$\frac{\delta}{x} \sim \frac{1}{\sqrt{Ux/v}} = \frac{1}{\sqrt{Re}}$$

(20%)

2. A long pipe is connected to a large open reservoir that is initially filled with water to a depth of $H$. The pipe is $D$ in diameter and $L$ long. As a first approximation, friction may be neglected. Determine the flow velocity leaving the pipe as a function of time after a cap is removed from its free end. The reservoir is large enough so that the change in its level may be neglected.

(25%)

Hint:

(1) assumption: velocity in reservoir may be neglected except for a small region near the inlet to the tube.

(2) Integrals of rational functions

$$\int \frac{1}{x^2 - a^2} \, dx = \begin{cases} \frac{1}{a} \arctanh \frac{x}{a} = \frac{1}{2a} \ln \frac{a-x}{a+x} + C & \text{for } |x| < |a| \\ \frac{1}{a} \text{arccoth} \frac{x}{a} = \frac{1}{2a} \ln \frac{x-a}{x+a} + C & \text{for } |x| > |a| \end{cases}$$
3. Let a closed loop of fluid particles \( C(t) \) be defined by

\[
X = a(\cos s + \alpha \sin s, \sin s, 0) \quad 0 \leq s \leq 2\pi
\]

where each value of \( s \) corresponds to a different fluid particle, and \( a, \alpha > 0 \).

Figure above is the \( C(t) \) at \( t=0 \) (circle) and \( t=t_1 \) (ellipse).

(a) Find out the Eulerian velocity field \( V(x,y,z,t) \) of this problem.
(b) Check if this Eulerian velocity field is a solution of Euler’s equation.
(c) Calculate the circulation at any time \( t \)
(d) Check if Kelvin circulation theory is satisfied.

(20%)

4. Show that for an incompressible fluid, the following identity holds between the velocity vector \( u \) and the vorticity vector \( \omega \).

\[
\nabla \cdot \left( (\bar{u} \cdot \nabla)\bar{u} \right) = \frac{1}{2} \left( \nabla^2 (\bar{u} \cdot \bar{u}) - \bar{u} \cdot (\nabla^2 \bar{u}) - \bar{\omega} \cdot \bar{\omega} \right)
\]

(20%)

5. In cylinder coordinates, the velocity components for uniform flow around a circular cylinder are

\[
u_R = U \left( 1 - \frac{a^2}{R^2} \right) \cos \theta \quad u_\theta = -U \left( 1 + \frac{a^2}{R^2} \right) \sin \theta
\]

Here \( U \) is the constant magnitude of the velocity approaching the cylinder and \( a \) is the radius of the cylinder.

(a) If compressible and viscous effects are negligible, determine pressure \( p(R, \theta) \) at any point in the fluid in the absence of any body forces. Take the pressure far from the cylinder to be constant and equal to \( P_0 \). (10%)
(b) Specialize the result obtained above to obtain an expression for the pressure \( p(a, \theta) \) on the surface of the cylinder. (10%)