Advanced Fluid Mechanics: Homework #5

1. Consider the two-dimensional flow field defined by the following velocity components: \( u = \frac{v}{1+t} \), \( v = 1 \), \( w = 0 \)
   
   For this flow field find the equation of the streakline at \( t=0 \) through the point \((1,1)\) (20%)

2. The velocity components for a particular flow field are as follows:
   \( u = 16x^2 + y \), \( v = 10 \), \( w = yz^2 \)

   (a) Determine the circulation, \( \Gamma \), for this flow field around the following contour by integrating the velocity around it:
   
   \[
   \begin{align*}
   0 \leq x \leq 10 & \quad y = 0 \\
   0 \leq y \leq 5 & \quad x = 10 \\
   0 \leq x \leq 10 & \quad y = 5 \\
   0 \leq y \leq 5 & \quad x = 0
   \end{align*}
   \]

   (b) Calculate the vorticity vector, \( \omega \), for the given flow field and hence evaluate:
   \[
   \int_A \omega \cdot n \, dA
   \]
   where \( A \) is the area of the rectangle defined in (a), and \( n \) is the unit normal to that area. Compare the result obtained in (b) with that obtained in (a). (15%)

3. (prob2.4 in textbook)

2.4. Consider the two-dimensional velocity distribution defined by:
   \[
   u = -\frac{x}{x^2 + y^2} \quad v = \frac{y}{x^2 + y^2}
   \]

   Determine the circulation for this flow field around the following contour by integrating the velocity around it:
   
   \[
   \begin{align*}
   -1 \leq x \leq +1 & \quad y = -1 \\
   -1 \leq y \leq +1 & \quad x = +1 \\
   -1 \leq x \leq +1 & \quad y = +1 \\
   -1 \leq y \leq +1 & \quad x = -1
   \end{align*}
   \]
   (20%)

4. The velocity components for a particular two-dimensional flow field are defined as follows:
   \[
   u = -\frac{y}{x^2 + y^2} \quad v = \frac{x}{x^2 + y^2}
   \]

   (a) Using the same contour as defined in Prob. 2.4, determine the circulation for the given flow field.
   (b) Calculate the vorticity vector for the given flow field.
   (c) Calculate the divergence of the velocity vector for the given flow field. (30%)
1. Consider the two-dimensional flow field defined by the following velocity components:

\[
u = \frac{v}{1+t}, \quad v = 1, \quad w = 0
\]

For this flow field find the equation of the streakline at \(t=0\) through the point \((1,1)\).

\[
\frac{dx}{dt} = u = \frac{v}{1+t} = \frac{1}{1+t}
\]

\[
\frac{dy}{dt} = v = 1
\]

\[
x = \ln(1 + t) + c_1
\]

\[
y = t + c_2
\]

\(t=\tau, \ x=1, \ y=1\)

\[
1 = \ln(1 + \tau) + c_1 \Rightarrow c_1 = 1 - \ln(1 + \tau)
\]

\[
x = 1 + \ln \frac{1 + t}{1 + \tau}
\]

\[
1 = \tau + c_2 \Rightarrow c_2 = 1 - \tau
\]

\[
y = t - \tau + 1
\]

\(t=0, \ x=1, \ y=1\)

\[
x = 1 - \ln(1 + \tau)
\]

\[
y = 1 - \tau \Rightarrow \tau = 1 - y
\]

The streakline is \(x = 1 - \ln(2 - y)\)
2. The velocity components for a particular flow field are as follows:
   \[ u = 16x^2 + y \quad v = 10 \quad w = yz^2 \]

(a) Determine the circulation, \( \Gamma \), for this flow field around the following contour by integrating the velocity around it:
   \[ 0 \leq x \leq 10 \quad y = 0 \]
   \[ 0 \leq y \leq 5 \quad x = 10 \]
   \[ 0 \leq x \leq 10 \quad y = 5 \]
   \[ 0 \leq y \leq 5 \quad x = 0 \]

   \[ a = (10, 0) \quad b = (0.5) \quad c = (10, 5) \]

   \[ \vec{V} = u \vec{i} + v \vec{j} + w \vec{k} = (16x^2 + y) \vec{i} + 10 \vec{j} + yz^2 \vec{k} \]

   \[ \Gamma = \oint_c \vec{V} \cdot d\vec{l} \]

   \[ \Gamma = \int_0^a \vec{V} \cdot dx \vec{i} + \int_a^c \vec{V} \cdot dy \vec{j} + \int_c^b \vec{V} \cdot dx \vec{i} + \int_b^0 \vec{V} \cdot dy \vec{j} \]

   \[ \Gamma = \int_0^{10} (16x^2 + y) \, dx + \int_0^5 10 \, dy + \int_0^{10} (16x^2 + y) \, dx + \int_5^0 10 \, dy \]

   \[ \Gamma = \int_0^{10} (16x^2 + 0) \, dx + \int_0^5 10 \, dy + \int_0^{10} (16x^2 + 5) \, dx + \int_5^0 10 \, dy \]

   \[ \Gamma = \left[ \frac{16}{3} x^3 \right]_0^{10} + [10y]^5_0 + \left[ \frac{16}{3} x^3 + 5x \right]_0^{10} + [10y]^5_0 \]

   \[ \Gamma = \frac{16}{3} (10)^3 + 50 - \frac{16}{3} (10)^3 - 50 - 50 \]

   \[ \Gamma = -50 \quad \text{(counter-clockwise)} \]

(b) Calculate the vorticity vector, \( \omega \), for the given flow field and hence evaluate:

   \[ \int_A \omega \cdot n \, dA \]

   where \( A \) is the area of the rectangle defined in (a), and \( n \) is the unit normal to that area.

   \textbf{Compare the result obtained in (b) with that obtained in (a).}

   \[ \omega = \nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 16x^2 + y & 10 & yz^2 \end{vmatrix} = z^2 \vec{i} - \vec{k} \]

   \[ \int_A \omega \cdot n \, dA = \int_A (z^2 \vec{i} - \vec{k}) \cdot \vec{k} \, dA \]

   \[ \int_A \omega \cdot n \, dA = \int_A -1 \, dA = -A = -50 \]

   The result is the same as (a), so it implies

   \[ \Gamma = \oint_c \vec{V} \cdot d\vec{l} = \int_A \omega \cdot n \, dA \]
3. Consider the two-dimensional velocity distribution defined by:

\[ u = -\frac{x}{x^2 + y^2} \quad v = \frac{y}{x^2 + y^2} \]

Determine the circulation for this flow field around the following contour by integrating the velocity around it:

\[ -1 \leq x \leq +1 \quad y = -1 \]
\[ -1 \leq y \leq +1 \quad x = +1 \]
\[ -1 \leq x \leq +1 \quad y = +1 \]
\[ -1 \leq y \leq +1 \quad x = -1 \]

\[ a=(1,1) \quad b=(-1,1) \quad c=(-1,-1) \quad d=(1,-1) \]

\[ \vec{V} = \left( -\frac{x}{x^2 + y^2} \right)\vec{i} + \left( \frac{y}{x^2 + y^2} \right)\vec{j} \]

\[ \Gamma = \oint \vec{V} \cdot d\vec{l} \]

\[ \Gamma = \int_a^b \vec{V} \cdot d\vec{x} + \int_b^c \vec{V} \cdot d\vec{y} + \int_c^d \vec{V} \cdot d\vec{x} + \int_d^a \vec{V} \cdot d\vec{y} \]

\[ \Gamma = \int_1^{-1} \left( -\frac{x}{x^2 + 1} \right)dx + \int_1^{-1} \left( \frac{y}{(-1)^2 + y^2} \right)dy + \int_{-1}^1 \left( -\frac{x}{x^2 + (-1)^2} \right)dx + \int_{-1}^1 \left( \frac{y}{1^2 + y^2} \right)dy \]

\[ \Gamma = -\int_1^{-1} \left( \frac{x}{x^2 + 1} \right)dx + \int_1^{-1} \left( \frac{y}{1 + y^2} \right)dy - \int_{-1}^1 \left( \frac{x}{x^2 + 1} \right)dx + \int_{-1}^1 \left( \frac{y}{1 + y^2} \right)dy \]

\[ \Gamma = \int_{-1}^1 \left( \frac{x}{x^2 + 1} \right)dx - \int_{-1}^1 \left( \frac{x}{x^2 + 1} \right)dx + \int_{-1}^1 \left( \frac{y}{1 + y^2} \right)dy - \int_{-1}^1 \left( \frac{y}{1 + y^2} \right)dy \]

\[ \Gamma = 0 \quad \text{(counter-clockwise)} \]
\[ u = \frac{-y}{x^2 + y^2}, \quad v = \frac{x}{x^2 + y^2} \]

\[ \vec{V} = \left( \frac{-y}{x^2 + y^2} \right) \hat{i} + \left( \frac{x}{x^2 + y^2} \right) \hat{j} \]

\[ d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k} \]

a) \[ \oint \vec{F} \cdot d\vec{r} = \int_{\Delta} \int_{\gamma} \nabla \cdot \vec{F} \, dA \]

\[ = \int_{\Delta} \left( \frac{x}{x^2 + y^2} \right) \, dx + \int_{\gamma} \left( \frac{-y}{x^2 + y^2} \right) \, dy + \int_{\Delta} \left( \frac{x}{x^2 + y^2} \right) \, dx + \int_{\gamma} \left( \frac{y}{x^2 + y^2} \right) \, dy \]

\[ = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} = 2\pi \]

b) \[ \vec{\omega} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{x}{x^2 + y^2} & \frac{y}{x^2 + y^2} & 0 \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} & 0 \end{vmatrix} \]

\[ = \left( \frac{y}{x^2 + y^2} \right) \hat{k} \]

\[ = \vec{c} \hat{k} \]

c) \[ \nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \]

\[ = \frac{\partial}{\partial x} \left( \frac{-y}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left( \frac{x}{x^2 + y^2} \right) \]

\[ = \left[ -\frac{y(2x)}{(x^2 + y^2)^2} \right] + \left[ \frac{-x(2y)}{(x^2 + y^2)^2} \right] = 0 \]