INTERFACE CRACKS/CORNERS IN ANISOTROPIC/PIEZOELECTRIC MATERIALS

Chyanbin Hwu and Tai-Liang Kuo
Institute of Aeronautics and Astronautics, National Cheng Kung University
Tainan, Taiwan, R.O.C.
Email: Chwu@mail.ncku.edu.tw

ABSTRACT
In order to have a detailed and thorough understanding about the failure behavior of elastic materials, several similar but different topics are usually studied separately, such as cracks versus corners, anisotropic materials versus piezoelectric materials, and cracks/corners lying in the homogeneous materials versus along the interfaces between two dissimilar materials. Since homogenous cracks and interface cracks are just special cases of interface corners, in order to build a direct connection among them a unified definition for their stress intensity factors was proposed in our recent study for general anisotropic elastic materials. Knowing that by using the Stroh’s complex variable formalism the solutions of problems with piezoelectric materials will preserve the same mathematical matrix form as those of the corresponding problems with anisotropic materials, the proposed unified definition was further extended to the piezoelectric materials. To have a direct feeling about their connection, same mathematical solution was applied to several different problems, such as the computation of singular orders and stress intensity factors for homogeneous/interface cracks/corners in anisotropic/piezoelectric materials. Through this work, several common features about the failure of materials can be observed. Hope that it will be helpful for the establishment of a universal failure criterion and fracture toughness.

INTRODUCTION
Due to the widely applications of anisotropic/piezoelectric materials and their possible failure induced by cracks or corners, several different kinds of interface corners/crack problems for anisotropic/piezoelectric materials have been discussed in the literature such as (Hwu, 1993; Suo, et al., 1992). By using the Stroh’s complex variable formalism (Ting, 1996), it’s known that the stress analysis for piezoelectric materials will preserve the same mathematical matrix form as those of the corresponding problems with anisotropic materials, in which the only difference comes from the expansion of matrix dimension to include the piezoeffects. To emphasize the common feature of the cracks and corners, interface and homogeneous, anisotropic and piezoelectric, in this paper we will follow the works of (Hwu and Kuo, 2007) and (Hwu and Ikeda, 2008) to present the computations of stress singular orders and stress intensity factors by using the same mathematical formulas for both anisotropic materials and piezoelectric materials.

COMPLEX VARIABLE FORMULATION
The basic equations for the stress analysis of two-dimensional linear anisotropic elastic solids include the stress-strain laws, strain-displacement relations and the equilibrium equations. These three equation sets constitute 15 partial differential equations with 15 unknown functions in terms of three coordinate variables. If only the two-dimensional deformation is
considered, the complex variable formulation can be used to establish the general solution for these 15 unknown functions satisfying 15 basic equations. In the literature, there are two different complex variable formulations for two-dimensional linear anisotropic elasticity. One is the Lekhnitskii formulation (Lekhnitskii, 1963, 1968) which starts with the equilibrated stress functions followed by compatibility equations, and the other is the Stroh formalism (Stroh, 1958, 1962; Ting, 1996) which starts with the compatible displacements followed by equilibrium equations. While the former is usually expressed in component form, the latter is commonly expressed in matrix form. Since the matrix can be adjusted or expanded to include appropriate material properties, when studying the piezoelectric materials, Stroh formalism has been purposely organized to keep the same matrix form as that of anisotropic materials. To keep this special feature, in this paper we like to present the general solutions satisfying all the basic equations for elasticity problems by using Stroh formalism. They are

\[
\begin{align*}
\mathbf{u} &= 2 \text{Re}\{\mathbf{Af}(z)\}, \quad \mathbf{\phi} = 2 \text{Re}\{\mathbf{Bf}(z)\},
\end{align*}
\]

(1)

where \( \mathbf{u} \) and \( \mathbf{\phi} \) are, respectively, the displacement vector and stress function vector; \( \mathbf{A} \) and \( \mathbf{B} \) are material eigenvector matrices; \( \mathbf{f}(z) \) is the complex function vector which are determined by the boundary conditions set for the problems; Re stands for the real part. For different kind of problems, these vectors or matrices will have different dimensions and different contents such as

**anisotropic materials:**

\[
\begin{align*}
\mathbf{u} &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \mathbf{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \quad \mathbf{f}(z) = \begin{bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \end{bmatrix},
\end{align*}
\]

(Aa)

\[
\begin{align*}
\mathbf{A} &= [\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3], \quad \mathbf{B} = [\mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3],
\end{align*}
\]

\[
\begin{align*}
z_k &= x_i + \mu_k x_2, \quad k = 1, 2, 3,
\end{align*}
\]

in which the stress functions \( \phi_i, i = 1, 2, 3 \) are related to the stresses by

\[
\begin{align*}
\sigma_{1i} &= -\phi_{i,2}, \quad \sigma_{2i} = \phi_{i,1}, \quad i = 1, 2, 3,
\end{align*}
\]

(2a)

**piezoelectric material:**

\[
\begin{align*}
\mathbf{u} &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad \mathbf{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}, \quad \mathbf{f}(z) = \begin{bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \\ f_4(z_4) \end{bmatrix},
\end{align*}
\]

(3a)

\[
\begin{align*}
\mathbf{A} &= [\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3 \mathbf{b}_4],
\end{align*}
\]

\[
\begin{align*}
z_k &= x_i + \mu_k x_2, \quad k = 1, 2, 3, 4,
\end{align*}
\]

in which \( u_i \) and \( \phi_i \) are related to the electric fields \( E_i \) and electric displacements \( D_i \) by

\[
\begin{align*}
u_{4,i} &= -E_j, \quad \phi_{4,i} = -D_i, \quad \phi_{4,i} = D_2.
\end{align*}
\]

(3b)
ORDERS OF STRESS/ELECTRIC SINGULARITY

Based upon the general solution shown in (1) and the boundary conditions set for the multi-bonded wedges, the orders of stress singularity can be determined by the following eigen-relation (Hwu, et al., 2003; Hwu and Lee, 2004), which is valid for cracks, interface cracks, corners and interface corners, and the materials can be any kinds of linear elastic anisotropic materials or piezoelectric materials.

\[
\begin{align*}
\text{bonded:} & \quad \|K_e - I\| = 0, \\
\text{free-free:} & \quad \|K_e^{(1)}\| = 0, \quad \text{fixed-fixed:} \quad \|K_e^{(2)}\| = 0, \\
\text{free-fixed:} & \quad \|K_e^{(3)}\| = 0, \quad \text{fixed-free:} \quad \|K_e^{(4)}\| = 0,
\end{align*}
\]

where \(K_e^{(i)}, i = 1, 2, 3, 4\) are the submatrices of \(K_e\) defined by

\[
K_e = \begin{bmatrix}
K_e^{(1)} & K_e^{(2)} \\
K_e^{(3)} & K_e^{(4)}
\end{bmatrix}, \quad K_e = \prod_{k=1}^{n} E_{n-k+1} = E_n E_{n-1} \cdots E_1,
\]

and

\[
E_k = \hat{N}^{1-\delta}(\theta_k, \theta_{k-1}), \quad k = 1, 2, 3, \ldots, n.
\]

\(\theta_k, \theta_{k-1}\) are the angular location of the two sides of the \(k\)th wedge, and \(\hat{N}\) is the key matrix defined by

\[
\hat{N}(\theta, \alpha) = \cos(\theta - \alpha)I + \sin(\theta - \alpha)N(\alpha),
\]

where \(I\) is a unit matrix and \(N(\alpha)\) is the generalized fundamental elasticity matrix of \(N\) (Ting, 1996). The \(\delta\) appeared in the power of key matrix \(\hat{N}\) in (4c) is the singular order to be determined, which may be positive or negative or zero, real or complex, repeated or distinct. If the stress singularity is concerned and the strain energy should be bounded, only the region \(0 < \text{Re}(\delta) < 1\) is considered. Note that the generalized fundamental elasticity matrix \(N(\alpha)\) is a 6×6 matrix for anisotropic materials and is a 8×8 matrix for piezoelectric materials.

STRESS/ELECTRIC INTENSITY FACTORS

A unified definition of stress intensity factors which correspond to the most critical singular order \(c\delta\) and are valid for cracks, interface cracks, corners and interface corners, and anisotropic and piezoelectric materials was proposed as (Hwu and Kuo, 2007; Hwu and Ikeda, 2008)

\[
k = \lim_{r \to 0} \sqrt{2\pi r^\delta_\varepsilon} \Lambda < (r / \ell)^{-\varepsilon_\varepsilon} > \Lambda^{-1} \phi_r (r, 0),
\]

where \((r, \theta)\) is the polar coordinate with origin located on the tip of cracks/corners; \(\delta_\varepsilon\) and \(\varepsilon_\varepsilon\) are, respectively, the real and imaginary part of the most critical singular order; \(\Lambda\) is a matrix related to the wedge configurations and properties; the angular bracket \(< \cdot >\) stands for a diagonal matrix in which each component is varied according to the subscript \(\alpha\); \(\ell\) is a length parameter which may be chosen arbitrarily as long as it is held fixed when specimens of a given material pair are compared; \(\phi\) is the stress function vector and the subscript comma
stands for differentiation. For anisotropic materials, the vector of stress intensity factors \( \mathbf{k} \) includes three different modes, i.e.,

\[
\mathbf{k} = \begin{cases} 
K_{II} \\
K_I \\
K_{III} 
\end{cases},
\]

whereas for piezoelectric materials, the additional electric intensity factor \( K_{IV} \) is added as the fourth component of \( \mathbf{k} \).

To provide a stable and efficient computing approach for the general mixed-mode stress intensity factors, the path-independent H-integral based on reciprocal theorem of Betti and Rayleigh was established in (Hwu and Kuo, 2007) for 2D problems and in (Kuo and Hwu, 2009) for 3D problems, i.e.,

\[
2D: \quad H = \int_{\Gamma} (\mathbf{u}^T \mathbf{t} - \hat{\mathbf{u}}^T \hat{\mathbf{t}}) d\Gamma,
\]

\[
3D: \quad H = \int_{\Gamma} (\mathbf{u}^T \mathbf{t} - \hat{\mathbf{u}}^T \hat{\mathbf{t}}) d\Gamma + \int_{\Sigma} (\hat{\sigma}_{i3}u_i + \hat{\sigma}_{i3}u_i,3 - \sigma_{i3}\hat{u}_i - \sigma_{i3}\hat{u}_i,3) dS,
\]

in which \( \mathbf{u} \) and \( \mathbf{t} \) are the displacement and traction vectors of the actual system, and \( \hat{\mathbf{u}} \) and \( \hat{\mathbf{t}} \) are those of the complementary system; the path \( \Gamma \) emanates from the lower wedge flank and terminates on the upper wedge flank in counterclockwise direction. For piezoelectric materials, the fourth component of traction vector \( \mathbf{t} \) is surface electric displacement and the fourth component of displacement vector \( \mathbf{u} \) is the electric potential.

By using the near tip solutions obtained in the literature (Hwu and Kuo, 2007), it has been proved that the stress intensity factor \( \mathbf{k} \) is related to the path-independent H-integral by

\[
\mathbf{k} = \sqrt{2\pi A} \mathbf{H}^* (1 - \delta_{K} + i\varepsilon_{\alpha}) \mathbf{e}_{\alpha}^{\mu} > \mathbf{H}^{-1} \mathbf{h},
\]

where \( \mathbf{H}^* \) is a matrix related to the near tip and complementary near tip solutions and \( \mathbf{h} \) is a vector consisting the value of H-integral calculated with specified complementary solution.

**NUMERICAL EXAMPLES**

To illustrate the applicability of the formulae listed in this paper to both corners and cracks, both anisotropic materials and piezoelectric materials, two typical examples are shown below.

**Example 1. An elliptical central crack in a homogeneous isotropic material.**

Theoretical study of this example has been firstly done by (Irwin, 1962) and the analytical solution was derived as

\[
K_I = \frac{\sigma_0 \sqrt{\pi b}}{E(k)} (1 - k^2 \cos^2 \varphi)^{1/4},
\]

\[
E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{1/4} d\theta, \quad k^2 = 1 - \left( \frac{b}{a} \right)^2.
\]

in which the parameters \( a \) and \( b \) are the half lengths of the major and minor axes of the elliptical crack and are prescribed to be \( a=25\text{mm} \) and \( b=10\text{mm} \); \( \sigma_0 \) is the remote tension specified as 10MPa (Figure 1); \( E(k) \) is the elliptical integral. The elliptical central crack is
embedded in an isotropic cylinder whose section radius and half longitudinal length are both equal to $10a$, and whose Young's modulus and Poisson's ratio are 150GPa and 0.3. When performing the finite element analysis, the boundary conditions are set to be $u_i = 0$ on the surface $x_1 = 0$, $u_3 = 0$ on the surface $x_3 = 0$, and $u_2 = 0$ on the circular line $\rho = 10a$ & $x_2 = 0$ due to the symmetry of the problem.

This example is a conventional 3D crack problem whose singular order is a well-known number 0.5 which is a real triple root with three independent eigen-functions. Through the relation, eqn. (8), it can be calculated that

$$k = \frac{\sqrt{2\pi}}{2} \begin{bmatrix} 0 & \lambda_{31} & 0 \\ \lambda_{12} & 0 & 0 \\ 0 & 0 & \lambda_{33} \end{bmatrix} \begin{bmatrix} H_{11}^* & H_{12}^* & H_{13}^* \\ H_{21}^* & H_{22}^* & H_{23}^* \\ H_{31}^* & H_{32}^* & H_{33}^* \end{bmatrix}^{-1} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \begin{bmatrix} K_{II} \\ K_{III} \end{bmatrix}, \tag{10}$$

which shows that in general no mode of the stress intensity factors will be lost if only the most critical singular order is considered, and hence it is not necessary to consider the stress intensity factors for the lower singular orders. For the present pure tension case $H_1 \neq 0$, $H_2 = H_3 = 0$, and consequently only $K_I \neq 0$ will be determined in this example. Therefore, in this case $K_{II} = K_{III} = 0$ does not mean that these two modes are lost due to the consideration of the most critical singular order. They are zero totally due to the pure tension condition. Note that the effects of the external loading condition on the stress intensity factors are reflected through $h$ of eqn. (10), and are nothing to do with $\Lambda$ and $H^*$.

Table 1 shows the results of $K_I$ versus $\phi$ calculated by the analytical solution, eqn. (9), as well as the present method through four different domains. The results prove the property of domain-independency and the accuracy of the present method since the maximum relative error is below 0.5% which is acceptable.

Since the domain-independency has been proved theoretically and numerically, in the following example the numerical data will be presented only for one domain-integral and the comparison with the other different domain integrals will not be presented to save the space of this paper.

Example 2: An interface corner between two dissimilar piezoelectric ceramics.

The interface corners between two dissimilar piezoelectric ceramics are analyzed to obtain the orders of stress/electric singularity and stress/electric intensity factors (Figure 2). The materials below and above the interface are PZT-5H and PZT-7A, which are both polarized along $x_2$-axis with the material constants listed in Table 2. The two outer surfaces of the corners are both assumed to be impermeable. In our study the corner angles vary within $0^\circ$-$160^\circ$ in which $0^\circ$ represents interface crack. When employing the H-integral to calculate the stress/electric intensity factors, the displacements, stresses, electric displacements, and electric potentials of the actual system are calculated via finite element analysis. The boundary conditions used in finite element analysis are: $u_i = 0$ on the edge $x_i = -a$, $u_2 = 0$ on the point $x_1 = W$ & $x_2 = 0$, $\sigma_0 = 1$MPa and $u_4 = 0$ on the edge $x_3 = W$, while $\sigma_0 = 1$MPa and $D_0 = -0.001C/m^2$ on the edge $x_2 = -W$. Table 3 shows the most critical orders of stress/electric singularity $\delta_e$ and their associated stress/electric intensity factors versus notch
half angle $\beta$. Since no analytical solution of the stress/electric intensity factors has been presented in the literature, the comparison is made only with the existing solutions for the interface cracks, which shows that our results agree well with the reference solutions for interface crack.

CONCLUSIONS
A universal formula for the singular orders and stress intensity factors, eqs. (4) and (5), is proposed, which is valid for cracks, interface cracks, corners and interface corners, and the materials can be any kinds of linear elastic anisotropic materials or piezoelectric materials.

ACKNOWLEDGEMENTS
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REFERENCES
Table 1 Stress intensity factor $K_I$ versus the crack front location $\phi$ for an elliptical central crack in a homogeneous isotropic material subjected to remote tension.

<table>
<thead>
<tr>
<th>$\phi$ [degree]</th>
<th>Analytical sol., eqn. (9)</th>
<th>$K_I$ [MPa·mm$^{0.5}$]</th>
<th>H-integral</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>domain_1 ($r=0.08a$)</td>
<td>domain_2 ($r=0.10a$)</td>
</tr>
<tr>
<td>3.6497</td>
<td>30.9703</td>
<td>31.1267</td>
<td>31.1144</td>
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<td>17.0800</td>
<td>33.8234</td>
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</tr>
<tr>
<td>31.6885</td>
<td>38.5383</td>
<td>38.6444</td>
<td>38.6282</td>
</tr>
<tr>
<td>41.4324</td>
<td>41.5196</td>
<td>41.6266</td>
<td>41.6080</td>
</tr>
<tr>
<td>50.9173</td>
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<td>44.1210</td>
</tr>
<tr>
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</tr>
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<td>59.5164</td>
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<td>67.5469</td>
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<td>47.3715</td>
<td>47.3506</td>
</tr>
<tr>
<td>75.2122</td>
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<td>48.2719</td>
<td>48.2505</td>
</tr>
<tr>
<td>82.6578</td>
<td>48.5434</td>
<td>48.6993</td>
<td>48.6780</td>
</tr>
</tbody>
</table>
Fig. 2 An interface corner lying along the interface between two diverse piezoelectric materials.

Table 2 Material constants of PZT-5H and PZT-7A.

<table>
<thead>
<tr>
<th></th>
<th>PZT-5H</th>
<th>PZT-7A</th>
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</thead>
<tbody>
<tr>
<td>$C_{11}, C_{33}$ [GPa]</td>
<td>126</td>
<td>148</td>
</tr>
<tr>
<td>$C_{12}, C_{31}$ [GPa]</td>
<td>53</td>
<td>74.2</td>
</tr>
<tr>
<td>$C_{13}$ [GPa]</td>
<td>55</td>
<td>76.2</td>
</tr>
<tr>
<td>$C_{22}$ [GPa]</td>
<td>117</td>
<td>131</td>
</tr>
<tr>
<td>$C_{44}, C_{66}$ [GPa]</td>
<td>35.3</td>
<td>25.4</td>
</tr>
<tr>
<td>$C_{55}$ [GPa]</td>
<td>35.5</td>
<td>55.9</td>
</tr>
<tr>
<td>$e_{31}$ [C/m²]</td>
<td>-6.5</td>
<td>-2.1</td>
</tr>
<tr>
<td>$e_{32}$ [C/m²]</td>
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<td>9.5</td>
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<td>$e_{33}$ [C/m²]</td>
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<td>$e_{16}, e_{34}$ [C/m²]</td>
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</tr>
<tr>
<td>$\varepsilon_{21, 33}$ [$10^9$ C/(V m)]</td>
<td>15.1</td>
<td>8.11</td>
</tr>
<tr>
<td>$\varepsilon_{23}$ [$10^9$ C/(V m)]</td>
<td>13</td>
<td>7.35</td>
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</table>
Table 3 Orders of stress/electric singularity and stress/electric intensity factors versus different values of $\beta$.

<table>
<thead>
<tr>
<th>$\beta$ [degree]</th>
<th>$a$ [mm]</th>
<th>$h$ [mm]</th>
<th>$W$ [mm]</th>
<th>$\delta_c$</th>
<th>$K_I$ [MPa m$^{\frac{1}{2}}$]</th>
<th>$K_{II}$ [MPa m$^{\frac{1}{2}}$]</th>
<th>$K_{IV}$ [(C/m$^2$) m$^{\frac{1}{2}}$]</th>
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<tr>
<td>0</td>
<td>10</td>
<td>0</td>
<td>300</td>
<td>0.5 ± 0.00697i ± 0.5</td>
<td>0.17747</td>
<td>-0.01634</td>
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<tr>
<td>10</td>
<td>10</td>
<td>1.76</td>
<td>300</td>
<td>0.49222</td>
<td>0.14733</td>
<td>0.06781</td>
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<tr>
<td>20</td>
<td>10</td>
<td>3.64</td>
<td>300</td>
<td>0.49412</td>
<td>0.19022</td>
<td>0.03318</td>
<td>4.0677e-5</td>
</tr>
<tr>
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<td>10</td>
<td>5.77</td>
<td>300</td>
<td>0.48587</td>
<td>0.21832</td>
<td>0.02538</td>
<td>4.2568e-5</td>
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<tr>
<td>40</td>
<td>10</td>
<td>8.39</td>
<td>300</td>
<td>0.46707</td>
<td>0.26466</td>
<td>0.04255</td>
<td>4.9274e-5</td>
</tr>
<tr>
<td>50</td>
<td>8.39</td>
<td>10</td>
<td>300</td>
<td>0.43351</td>
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<td>0.02417</td>
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</tr>
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<td>60</td>
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<td>300</td>
<td>0.37966</td>
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<td>0.04024</td>
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<td>80</td>
<td>1.76</td>
<td>10</td>
<td>300</td>
<td>0.17948</td>
<td>1.14229</td>
<td>0.07706</td>
<td>1.9262e-4</td>
</tr>
</tbody>
</table>

Note: 1 The number in the parentheses of $\delta_c$ is the reference solutions presented by (Ou and Wu, 2000).

2 The numbers in the parentheses of $K_I$, $K_{II}$, and $K_{IV}$ are the reference solutions presented by (Hwu and Ikeda, 2008).